THE MATHEMATICAL THEORY OF A NEW RELATIVITY

(Chapters VI to IX)

BY

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CHAPTER VI

Criticisms Answered

Section I

INTRODUCTION

Considerable misapprehension has been unfortunately aroused that the existence of gravitons is the essential foundation of my Theory. Although for mathematical facilities attention was concentrated in Chapter I on the path of a particle, it was clearly pointed out on page 5, paragraph 2, that the assumption of the existence of gravitons was not at all necessary for the theory. The same point was again emphasised on pages 219, 229 and 261 in the later chapters. But owing to the frequent reference to gravitons, the main idea was somewhat obscured. No doubt the *Physical Theory* involved the assumption that gravitational influence is propagated outwards in all directions equally, and thus necessarily led to a diminution of intensity in the ratio of the inverse square of the distance. Again if the attractive influence emanates from every mass, then it must necessarily be proportional to the mass of the influencing body, and if the influence is mutual, then it must also be proportional to the mass of the influenced body. But the only solitary assumption which is necessary for my Mathematical Theory is that, quite regardless of any physical

theory of gravitation and quite irrespective of the way in which gravitational attraction might be caused, *the gravitational influence is propagated on all sides from a body of mass outwards with a finite velocity.* From this one single assumption the rest of the results can be deduced purely mathematically. That the influence of gravitation should be propagated outwards from the Sun is a natural assumption, for if the influence were not propagated from the Sun towards the planet, but in the opposite direction, such influence would never reach the planet at all. The finiteness of the velocity of propagation necessitates the compounding of the velocities, producing the well-known aberration effect. The finiteness of the velocity also involves a change in frequency if the two bodies are relatively in motion. These necessary results are in themselves quite sufficient for the mathematical analysis in Chapters I and II.

Similarly, in Chapter III the assumption that light consists of swarms of rotating particles was quite sufficient to explain that a part of the apparent recessional velocity of nebulæ must be spurious, as light would be reddened as it passes through space. Even for Chapter IV it was not necessary to assume the existence of gravitons. Section V in particular avoided such an assumption. All that was necessary for this chapter was that some fine particles, maybe light corpuscles only or even heavier particles like electrons and even atoms and molecules, are thrown off from an outer layer in all directions. The effect of the motion of a nebula could then be sufficient to produce a difference in the losses of momenta in the front and at the rear, resulting in a net acceleration in the direction of motion, which will explain the recessional velocity proportional to distance after a sufficient distance has been covered.

As is observed for light, it was assumed that these material particles emerge from a moving nebula with constant velocity in space, that is to say, their spacial velocity is independent of the velocity of the source; and a physical explanation of it was also offered. This is just the assumption that is made in Relativity, without the additional assumption that this constant velocity in space is also a constant relative velocity, no matter howsoever fast bodies may be moving. But even this assumption is not essential for the theory. All that is necessary to assume is that particles that emerge are *not wholly* carried along by their source, that is to say, they do not in addition to their own intrinsic velocity possess the whole of the velocity of the source. So long as there is a residual difference between the velocity with which they are partially carried along and the velocity of the source, there would be a balance of momentum left which would result in a net acceleration in the

direction of motion. Thus by merely denying that light particles are wholly carried along by their source, one can arrive at the same mathematical result as in Section V. The constant co-efficient would then be of a smaller value, helping to prolong the ages of the nebulæ, and making the life of our galactic system longer than it appears, thus satisfying the needs of evolution better.

Four more chapters are submitted now. In Chapter VI certain published criticisms have been answered. Hamilton's supposition that my theory was similar to Lesage's or that it must founder on the same rock as Laplace's theory, has been refuted; and it is pointed out how the theory steers clear of that rock. It is claimed that this theory is the only known theory which can explain the positive value for increases in the eccentricities of the Earth, Mars and Venus, as observed by Newcomb. Some reasons have been given for not accepting Satyendra Ray's suggestion. A short note has been added on A. C. Banerji's remarks, and sundry other points also have been noted.

In Chapter VII the claim of Relativity that only relative velocities can be measured, and not absolute velocities, is examined, and it is shown that relative velocity as measured has no meaning, unless it is also specified how measured, as different methods of measuring relative velocities give entirely different results. It is just as impossible to measure exactly the actual relative velocity, *i.e.*, the difference between two absolute velocities, as the absolute velocities themselves. Newton's value is true to the first order term, and Einstein's to the second order terms. For higher approximations the relative velocity is a function of the actual velocities and not only of their difference.

Chapter VIII deals with the retardation of gravitational influence and points out certain further corrections which must be made to the Newtonian law of gravitation for more accurate results.

Chapter IX deals with the effect of a resisting medium, which would greatly nullify the effect of the accelerated motion and thereby reduce the perturbations.

The Appendix points out that the special shift in the light from the centre of the Sun's disk is unreliable, and a different value is calculated for the shift in the light from the limb.

It will appear that Newton's principles have not failed, but that his Mechanics requires several necessary corrections, which make the equations of motion highly complex, yielding more accurate results with higher approximations. On the other hand, the assumptions in Einstein's Relativity are mere approximations, and give tolerably good

results in practice, but not being rigorously true, the philosophy based thereon is a failure, making its results unintelligible to a three-dimensional being.

I must again express my gratitude to Mr. A. N. Chatterji, M.Sc., for his great kindness in helping me in checking the mathematical processes and to Dr. D. S. Kothari, D.Sc., Ph.D., for making valuable suggestions.

Section II

D. R. HAMILTON'S CRITICISMS

D. R. Hamilton, professedly with the "assistance" of H. P. Robertson, of the Princeton University, to which Einstein himself now belongs, has published a criticism of my Theory.[1] The criticism is not directed against anything published by me, but suggests certain untoward results that he supposes would follow from the finiteness of the velocity of gravitation in my Theory. I propose to meet his objections *seriatim* :—

1. He thinks that I have based my theory of gravitation on "gravitons."

As pointed out in the preceding section, this is a clear misapprehension so far as my Mathematical Theory is concerned. I maintain that I have made only *one solitary assumption and no more, viz., that the influence of gravitation travels with a finite and not infinite velocity.* My equations are mere mathematical deductions from this one single assumption.

2. He considers that my hypothesis of gravitons "is essentially the same as that put forward by Le Sage in 1764 as an explanation of gravitation."

Now my physical theory of gravitation is a theory of *emission* of matter and not of impinging or absorption.[2]

On the other hand, Le Sage's theory was that space is full of small corpuscles moving in all directions which hit material bodies and are thrown back.[3] Accordingly, one body partially shields another body, causing a net resultant effect of attraction. But the force, although inversely proportional to the square of the distance, would not be proportional to the masses, but would instead depend on the dimensions of the bodies. Clerk Maxwell showed that even the inference of the proportionality to dimensions was incorrect as although some corpuscles would

be intercepted by one body, others after reflection from it would hit the other body which would not otherwise do so. As suggested by Lorentz, the theory can be somewhat saved by further assuming that the corpuscles are wholly or partially absorbed by matter, but then the theory loses its simplicity.[4]

There is hardly any identity between the two. Further, as the existence of gravitons is not at all necessary for my Mathematical Theory, no question of any similarity with Le Sage's theory arises. The analogy disappears altogether.

3. He considers that my theory does not remove the objection to "Laplace's mathematically analogous theory."

Laplace's theory[5] was based on the idea that gravitational attraction was produced by the impulse of a fluid on the centre of the attracted body. No doubt he considered the possibility of a finite velocity of gravitation, but his theory led to the propagation of gravitational influence being directed from the influenced body to the influencing body, with the result that the extra-tangential force caused a *retardation* of motion. He started with the simple Newtonian law, and not only neglected the small deficiency in the component along the radius vector, but also quite arbitrarily assumed that the surplus component along the tangent was nearly the same as that along the transversal. These assumptions made the integrations quite easy. But his result of a retarded tangential motion, although it had no effect on the longitude of the perihelion, introduced large secular perturbations in the semi-major axis, the eccentricity and the mean longitude of the planet in the orbit, which would not tally with observations unless the velocity of gravitation were 6×10^6 times that of light in the case of Mercury. The idea of any velocity commensurable with that of light had therefore perforce to be abandoned. Neither Laplace nor Tisserand could think of a resisting medium as an explanation, because the equations led to a retardation, and a resisting medium would have made the result much worse. Laplace's equations did not yield any formula tallying with the observed advance of the perihelion of a planet. The gravitational deflection of light and the red shift of the spectral lines were not considered by them at all, and indeed were not known at the time.

My equations are derived from the analogy of the well-known principle of retarded potential in an electric attraction; and the gravitational influence is propagated naturally from the influencing body outward to the influenced body. The equations take account of the reduced effect due to the relative motion of the bodies and so contain additional

terms, which render the equations extremely complex and incapable of exact solution. Only a few solutions by successive approximations, obtained by omitting higher powers, have been so far published by me. But my equations definitely lead to an *acceleration* and not retardation of motion.

4. He acknowledges that the changed magnitude of the attractive force as deduced by me has "the net result of introducing an advance of the perihelion close to the desired value in the case of Mercury." It is also gratifying that there are no adverse remarks regarding my formulæ for the deflection of light and the spectral shift. These three are the chief results on which the claims of General Relativity are based.

5. He directs his criticisms against the resultant perturbations in the semi-major axis and the eccentricity only. Thus the results arrived at by me in Chapter I, much less those in Chapter II, are not challenged; and a premature attack is delivered in the form of an objection made in advance to what is to follow in a later chapter.

6. With regard to the perturbations in the semi-major axis and eccentricity, he suggests that I "had apparently not carried out the calculations." This remark is astonishing as I had at considerable length worked out the exact values of these perturbations in Chapter I, Section XII, pp. 20—24. Indeed in finding the values of these perturbations I had even ventured to claim an originality of treatment by saying on p. 24: "The method of treating the disturbed elliptic motion as being due to small additional forces which has been followed in this section is not known to have been adopted previously." The reason why the further consideration of these perturbations was intentionally postponed to a later chapter was that I was first concerned with offering substitutes for the three main formulæ of General Relativity, which it was commonly believed could not be deduced from Newtonian Mechanics.

7. He says that the absurd size of my perturbations can be realized if the cumulative effect of yearly increases in Mercury's eccentricity be considered.

Simple calculations based on yearly increases for a large number of revolutions are obviously fallacious. The solutions of my equations have been obtained by me after eliminating the time t. The curve thus found represents a geometrical picture of the orbit, and that too only within the limits of the approximations. It does not at all profess to give the whole history of the revolutions of a planet for a long period. The fallacy underlying the contrary assumption becomes at once apparent when similar increases are applied to Einstein's equations (8·2)

and (8·3) on page 12 which would then assume that $\cos \epsilon = 1$ and $\sin \epsilon = \epsilon = \frac{3\mu^2}{c^2 h^2} (2n\pi + \theta)$. These break down when n is large. The obvious explanation is that the geometrical shape of the orbit is one thing, and the whole history through its spiral path is quite another.

8. Quoting Laplace, he remarks "the resulting tangential acceleration (retardation ?) of the planets had no effect on the longitude of perihelion, but introduced secular perturbations in the semi-major axis and eccentricity of the orbit and in the mean longitude of the planet in the orbit." On the authority of Jenneck and Chazy, he concludes that the velocity of gravitation would have to be extremely large ranging from 6×10^4 times the velocity of light in the case of Mars, to 2×10^6 times in the case of the Earth. He argues that otherwise Mercury would leave the solar system in about 300 years from now, and that if the velocity be not so large, the advance of perihelion would be negligible.

It will be shown later that the force of gravitation along the shifted direction has a large component, nearly equal to Newton's value, along the radius vector, which would account for Kepler's perfect ellipse ; and it has a further small component along the radius vector and another slightly larger component, of the order of about 1/1000 in the case of the earth, transversely to the radius vector. These last two can be resolved into components along the tangent and the normal respectively. The normal component causes only periodic fluctuations, its net result on the semi-major axis and the eccentricity for a complete revolution is nil. It is only the tangential component, which causes acceleration of motion, tending to increase the semi-major axis and the eccentricity. Now Laplace and Tisserand had no explanation for the perturbations caused by the tangential retardation except the largeness of the velocity of gravitation which broke down the theory. But my Theory leads to a tangential acceleration and not retardation ; the tangential acceleration can be counteracted by the retardation caused by the resisting medium, which acts tangentially in the opposite direction, and while producing a null effect on the advance of the perihelion reduces the dreaded perturbations.

9. He admits that the sign predicted by my formula in the case of the eccentricities of Venus, Earth and Mars agrees with Newcomb's observations, but points out that the discrepancy in Mercury's eccentricity between Newtonian theory and observations is of opposite sign.

Now Laplace's theory could not offer any explanation of such a change of sign. And neither Newton's nor Einstein's theory can explain

an increase of semi-major axis and eccentricity for Venus, Earth and Mars, nor a decrease in the case of Mercury. But in my Theory, the effect of resistance being small for these larger planets, the sign remains unaltered; but as Mercury is closer to the Sun and the resistance larger, the retardation due to the resistance just overcomes the accelerations of motion, and the discrepancy changes sign. In the case of Encke's comet, which passes closer still, the resistance is so much that it actually causes a marked shortening of the period.

He confidently asserts that my Theory, "in so far as it relates to gravitation, would seem, then, to founder on the same rock as Laplace's mathematically analogous theory."

The *acceleration* produced in my Theory, in direct contrast with the *retardation* produced in Laplace's theory, makes an important difference, which saves my Theory from that fate.

10. D. R. Hamilton is obviously not aware that it has been noticed by Lorentz[6] and A. S. Eddington[7] that Laplace was quite wrong in concluding that unless the velocity of gravitation was enormously large, the tangential component would cause much larger perturbations than are actually observed. Indeed, on Laplace's own theory the effect of the first order term would be wholly compensated even if the velocity of gravitation is equal to the velocity of light, provided the retarded potential is taken into account. The difficulty will, however, remain that Laplace's theory would still fail to give the value for the advance of the perihelion, nor would it explain the spectral shift of light from the Sun.

11. I may here add that $-\frac{\mu}{r^2}-\frac{3\mu h^2}{D^2}\frac{1}{r^2}$ where h is twice the area described per second even for light and $D=c$, is an empirical law of gravitation which gives all the required results without causing any difficulty. It causes no perturbations in the major axis and the eccentricity, produces an advance of perihelion $=\frac{6\pi\mu^2}{D^2h^2}$ and the spectral shift from the centre $=\frac{\mu}{a}$. The deflection of light $=-\frac{\mu}{c^2R}\int_{-\pi/2}^{\pi/2}(1+3\cos^2\theta)\,d\theta$ $=-\frac{2\mu}{c^2R}\left(1+\frac{\pi}{2}\right)=2''\cdot 24$ tallying with Freundlich's mean value $2''\cdot 20\pm 10$.

SECTION III

SATYENDRA RAY'S SUGGESTION

1. Some time after the publication of the first two chapters of my Theory, Satyendra Ray of the Lucknow University suggested to me that

(*a*) instead of assuming that matter is emitting gravitons, I might assume that it is absorbing them and (*b*) that the algebraic quantity D in my equations, instead of being taken as of positive sign, might be taken to have a negative sign. (*c*). He thought that this change of sign would not materially alter the results, as it is only D^2 and not D that occurs in the main formulæ. (*d*) Before the publication of Chapters III and IV, when I had explained to him only "orally and summarily" the acceleration produced on an isolated moving nebula by its own emission, he inferred that even on my Theory "a moving body uniformly radiating in every direction experiences a force tending to stop its motion. . . . The motion itself generates a friction of motion which tends to stop it," in conformity with classical astronomers like Lagrange.

(1) The idea that gravitons are flying about in space and enter into and are absorbed by matter is exactly identical with that of Le Sage, as modified by Lorentz, already discussed in the preceding section. It is therefore not a new idea. The first objections to it are the same as there pointed out. Other objections are that if gravitons are emitted by matter at intervals and have the same velocity, they can travel with fixed spacing; but if they already exist in space at random, they cannot have any uniform spacing; also the theory involves a flow of matter from a lower material potential to a higher potential where there is already concentration, contrary to what we actually observe in radiation.

(2) The idea that D should be given a negative sign and therefore measured in the opposite direction is identical with Laplace's theory discussed in the preceding section, and is also not a new idea. The theory broke down because the large secular perturbations in the semi-major axis, the eccentricity and the mean longitude of the planet could not be explained by Laplace. The large decreases in these elements, which that theory produced, are not at all observed. As D. R. Hamilton has pointed out, if to get over this difficulty the value of D is increased sufficiently, the advance of perihelion would, contrary to observation, be negligible. Ray cannot therefore avoid the horns of Hamilton's dilemma. If the effect of the retarded potential is taken into account, then the value of the advance of the perihelion is very largely diminished.

(3) It is not quite accurate to say that only the second power of D occurs in the principal equations and therefore it matters little whether D is positive or negative. A negative D would alter the sign throughout.

If the propagation is in the direction opposite to v, the factor on p. 8 will be $(1+v/D)^3$ and the sign of D would be changed both in (5·4)

and (5·6). Therefore there would be a change in the value of the advance of the perihelion in (11·13). It would also alter the sign of $\triangle a$ in (12·2) and (12·3) and therefore also in (12·4); and similarly of $\triangle e$ in (12·6) and (12·7) and therefore also in (12·8) in which the first power of D occurs. These give decreases of the order 10^{-4}, whereas increases of the order 10^{-8} are mostly observed, falsifying a negative D.

(4) After the publication of Chapter IV, the last point vanishes. The effect is there shown to be an acceleration and not retardation. But he still maintains that γ in (20·1) may be negative. If so, the equation would produce a shrinking universe instead of an expanding one, which is contrary to observation. The utility of a theory lies in its concordance with observation ; if that does not exist the theory is useless.

(5) The conception of æther particles flying about in space, hitting protons and being absorbed by them was put forward by Tombrock of Holland in his Chemische Stofferklarung (1933). Gravitation was assumed to be caused by pressure of æther particles from all sides. A single body is held in equilibrium, but the presence of another body acts as a screen. The difference in the momenta on the outer and the inner sides of a body will result in motion towards the second body, appearing as gravitation. As pointed out by me in my Introduction to the English translation of his book by Jackson, the theory would lead to Newton's law of gravitation. The objection to it is that like Laplace's theory it leads to a velocity of gravitation in the opposite direction from the planet to the Sun causing the same difficulties.

2. The reasons why at present I cannot accept Ray's suggestion (already shown to be the same as that of Le Sage, Lorentz and Laplace) may be summarised as follows :—

(*a*) My mathematical theory is really independent of any physical theory of gravitation, and holds irrespective of the question whether gravitational effect is the result of emission or absorption of gravitons. I wish to keep my mathematical theory clear of any such commitment, and have based it simply on the propagation of gravitational influence from a body outward in all directions with a finite velocity.

(*b*) It is illogical to assume that the influence of gravitation is propagated from the influenced body towards the influencing body. If the influence of the Sun were propagated in a direction from the planet to the Sun, it will never reach the planet at all.

(*c*) D in my equations is an algebraical quantity and, mathematically speaking, can be both positive and negative. But I must reject a negative D, as that would lead to unexplained large secular perturbations

in the elements of the orbit, which are falsified by actual observations, both as to their sign as well as their magnitudes in the case of Venus, Earth and Mars.

(*d*) When the rotation of both the Sun and a planet are taken into account, the retarded gravitation would make the advance of the perihelion vanish almost completely.

(*e*) Laplace's theory curiously made the intensity of gravitational influence increase if a body were moving away from it, so that the value of the spectral shift on that view would be several times that of Einstein's.

(*f*) A negative D would make the nebulæ retard their motion, and would not explain their recession with a velocity proportional to their distances. What we would then have observed would be a shrinking and not an expanding universe.

I may add that if a negative D were found to accord better with observation, I would be only too happy to accept it. Although I cannot accept the idea that the influence of the gravitation of the Sun should have a velocity in the opposite direction, travelling from the planet towards the Sun, I would, *provided* it were not only dynamically sound, but also made to yield values of the advance of the perihelion and of the spectral shift tallying with observation, be prepared to adopt a modification of Laplace's theory, that even with a positive D, the gravitational effect on a moving body is obstructive, causing a slight retardation.

SECTION IV

A. C. BANERJI'S REMARKS

1. A. C. Banerji of the Allahabad University[8] has published some general remarks on my Theory. I am grateful to him for his "congratulating me on the accuracy in the mathematical working out of my Theory in which he could not find any flaw." I fully accept the tests which he has laid down for purposes of comparison, and claim that my Theory can stand them.

The Introduction explains that I have made only one solitary new assumption, *viz.*, the finiteness of the velocity of gravitation. His main criticism is that the emission of gravitons with constant velocities wholly along the radii is an extraordinary assumption inasmuch as cross-radial momentum is not taken account of. The answer is five-fold.

(1) The approximate constancy of the velocity of gravitons in space is exactly identical with the similar constancy of the velocity of light assumed in Relativity. The one is no more extraordinary than the other is. (2) The assumption that gravitons when they emerge have nearly a constant velocity necessarily involves the inference that such velocity is to some extent independent of the velocity of the source. This again is nothing stranger than what is assumed in Relativity for light. If the velocity is independent of the velocity of the source, the cross-radial momentum cannot be produced. Even if it is dependent, but only partially, there would be a net difference in the losses of momenta at the rear and in the front. (3) The assumption that the emissions are radial is the resultant average effect of the emissions from all particles in all directions. It is of course not the fact that the whole mass is really condensed at the centre with radial emissions from it; the practical result is as if the whole mass were so concentrated and the emissions were radial. From the symmetrical shape of a hemisphere, it will be apparent that the emissions from all corresponding points normal to its motion would balance one another, while their components along that direction will combine. (4) Radiation from the Sun alone means a loss of 250 million tons of mass per minute. Light is also known to exert pressure like moving material particles. And though it behaves like material particles, its velocity is not yet known to depend on the velocity of its source. (5) It was shown in Chapter IV that the existence of gravitons is not at all necessary, and the ordinary light corpuscles are quite sufficient to explain the motion of the nebulæ.

2. (1) According to Newton's law of the conservation of momentum only when m and v are both constant,

$$\frac{d(m.v)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt} = 0 + 0 = 0.$$

But if the mass were somehow being annihilated, then if

$$\frac{d(m.v)}{dt} = 0, \text{ we shall have } \frac{dv}{dt} = -\frac{v}{m}\frac{dm}{dt} \neq 0,$$

i.e., a net acceleration when the mass is decreasing.

(2) In the new theory, de Sitter's test of the binary stars is taken as proof of the observed fact that the velocity of the light emitted from a moving star whether from the front or the back is nearly the same. It is then shown that if light consisted of swarms of material radions, the effect on the star would be an acceleration in the direction of motion.

If a system consisting of a big mass M and two tiny masses m at the front and in the rear were moving with velocity v, then its momentum will be $mv + Mv + mv$. Now if owing to internal action and reaction the tiny masses were thrown forward and backward with velocity c in space and M be moving with v', the momentum of the system will be $-mc + Mv' + mc$. Hence $v' = \left(1 + \frac{2m}{M}\right)v$, *i.e.*, there is necessarily an increase in the velocity of the remaining body M in the direction of its motion.

The new theory merely explains that when a body is moving and at the same time losing mass equally in all directions owing to internal action and reaction, then if it is stationary the losses would balance each other; but if the body be moving while losing mass, there would be less momentum lost in the front than at the rear, yielding a net acceleration.

(3) Except in the extreme case when light particles emerging from a nebula have velocity $c+v$ in the front and $c-v$ at the back, there must always be a difference in the losses of momenta causing a continual change in velocity. All that is necessary for the theory is that the velocity of emerging light in space should be $< c+v$ in front and $> c-v$ at the rear.

Section V

SUNDRY POINTS

1. An eminent physicist of London, whose name I cannot disclose without previous permission, suggested that the factor in (4·3) on p. 7 of my theory might be $(1 - v/D)^4$. The Appendix to Chapter I (pp. 259-60) was added in deference to this suggestion in order to elaborate the point.

2. Another critic doubted whether equation (24·9) on page 261 was right. The Doppler principle undoubtedly gives the frequency $\dfrac{1-\frac{v'}{D}}{1-\frac{u}{D}}$ (see Chapter VII, Sec. II (1). As regards the relative distance, obviously the single journey method determines the gravitational effect. For a continued length of time the effect is practically the same as if A were at rest and B were moving away from it with velocity $(v' - u)$. Hence the result is nearly the same as in (24·14).

3. The factor $\left(1-\frac{v}{D}\right)^3$ has been obtained on the assumption that there is an expansion similar to that of spherical waves propagated with a finite velocity D outwards equally in all directions. A modified physical theory can certainly yield a different power and can also produce a similar factor in the denominator. It will therefore be convenient to have a generalised form from which the exact power can afterwards be deduced by equating it to the observed value.

4. Let us assume as a generalised form $(1-v/D)^m \cdot (1+v/D)^n$. (25·1) where m and n are any integers, positive or negative. The factor then is $=\left(1-\frac{m-n}{D}\cdot v\right)$ nearly, where v/D is small. This will mean a substitution of $(m-n)$ for 3 in (5·41) and (5·61) on p. 9; and $(m-n)$ for 3 and $(m-n-1)$ for 2 in (5·8) and (5·9) on p. 10.

For a nearly circular orbit the formula (11·13) for the advance of perihelion would then become

$$\epsilon = \frac{m-n}{D^2} \cdot \frac{\mu^2}{h^2} \cdot \frac{\theta}{(1+k\theta)^2} \quad . \quad . \quad . \quad . \qquad (25·2)$$

Similarly, in (14·4) on p. 26, (assuming that it holds for light) the equation for the deflection of light, $(m-n)$, will be substituted for 3 and

$$\text{then } \frac{d^2u}{d\theta^2} + u = \frac{(m-n+1)}{c^2} \cdot u^2 \quad . \quad . \quad . \quad . \qquad (25·3)$$

Hence the deflection will be $\frac{2(m-n+1)}{3}$ times Newton's value.

Thus the effect of these changes will make no difference in the concordance with observation if

$$\frac{m-n}{D^2} = \frac{3}{c^2}, \; i.e., \; D = \sqrt{\frac{m-n}{3}} \cdot c.$$

As a particular case, if $(m-n) = 2$ the rate of the advance of the perihelion would be reduced to two-thirds ; while the minimum deflection of light would be equal to that in Einstein's Relativity.

5. One learned critic wondered whether my formulæ give values tallying with observation, and wanted a proof that $D=c$, nearly.

From (11·13) $\epsilon = \frac{6\pi\mu^2}{D^2h^2(1+2\pi k)^2} = \frac{6\pi\mu^2}{c^2h^2}\{1+2\times 3·1416\times 1·628\times 10^{-4}\}$

for Mercury if $D=c$.

$= \frac{6\pi\mu^2}{c^2h^2}$ nearly, just as in Relativity,

tallying with Newcomb's observation.

From (14·5) and (14·6) the deflection of light comes to

$= \frac{8}{3}$ times the Newtonian value if $D=c$, nearly

$=2''\cdot32$ tallying far more closely with observation than Einstein's value.

From (15.2) the spectral shift, if $D=c$, comes to ·00836 tallying with the Relativity value ·0084.

It is also clear that when $D=c$ satisfies three independent and different equations and makes them tally with observation, it is proved that the velocity of gravitation is nearly equal to that of light.

As against this, the position in Relativity is hopeless. With regard to the speed of the propagation of gravitation, Eddington remarks:—

"If co-ordinates are chosen so as to satisfy a certain condition which has no clear geometrical importance, the speed is that of light; if the co-ordinates are slightly different the speed is altogether different from that of light. The result stands or falls by the choice of co-ordinates." He expresses the opinion that "The speed of gravitation is quite definite; only the problem of determining it does not seem to have yet been tackled correctly." [9]

6. As to avoid confusion recourse will not in future be had to gravitons, the formula for Newton's Gravitational constant obtained on the physical theory of gravitons is here put down, but the proof is withheld:

$$F=\left(\frac{N^2a^2\mu D}{32t}\right)\frac{MM^1}{R^2} \qquad . \quad . \quad . \quad . \quad . \quad (25·4)$$

where n=the number of gravitons emitted per unit mass per unit time; N=number of gravitons emitted from a unit mass at a time; t=the time interval after which they are emitted; a=the radius of a graviton; μ=the mass of a graviton; D=the velocity of gravitons.

$$\text{Hence } G=\frac{N^2a^2\mu D}{32t} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25·5)$$

The dimension of $G=\frac{L^3}{T^2}$.

7. I am grateful to the Editor of *Nature*[10] for an appreciative review in which after referring to my Theory he remarked "In short, an attempt is made to give an alternative explanation for the whole range of phenomena usually adduced in support of Einstein's theory" and concluded "If it can stand the test of criticism, it will obviously be of

great importance." I need hardly say that considering the variety of matters dealt with, I cannot possibly hope that the whole paper is altogether flawless. If the fundamental ideas be acceptable and the theory prove to be on right lines, and the mistakes, if any, rectifiable, my labour would be amply rewarded.

8. I must also express my thankfulness to the Editor of *Science*[11] for encouraging reviews of my Theory, characterised as "A sane border line between classical mechanics of Sir Issac Newton and the newer concepts of Prof. Einstein", which have helped to draw the attention of eminent mathematicians and scientists to it.

9. My thanks are equally due to the Editor of the *Science News Letter* for giving publicity to my theory.[12]

CHAPTER VII

Observed Relative Velocity

INTRODUCTORY

Relativity takes pride in the conception that Newton's absolute velocity is impossible of measurement, whereas relative velocity as observed is a measurable quantity. But it was pointed out in Chapter V, Section I, para 5, pp. 241-2 that Einstein's rejection of Newton's absolute velocities has not improved things in any way, but has rather made relative velocity as observed an uncertain quantity, depending on the particular method of measurement chosen, and varying as the method is changed. Relative velocity (other than the difference between Newton's absolute velocities) would then cease to have any definite meaning and merely indicate something shown by a particular experiment, different experimenters using different experiments would arrive at entirely different results. It is just as impossible to measure exactly the actual relative velocity, *i.e.*, the real difference between two absolute velocities as to measure the absolute velocities separately.

SECTION I

THE SINGLE JOURNEY METHOD

1. In Relativity it is assumed that velocity of light is absolute, and that light takes the same time to go from one moving body A to another moving body B no matter how differently the two bodies be moving. Einstein defines common time between A and B by quite arbitrarily assuming that "the time which light requires in travelling from A to B is equivalent to the time which light requires in travelling

from B to A."[13] This assumption is an utter physical impossibility when the two velocities are different, and can be true only in the case where the two bodies are moving with exactly equal velocities, either both towards each other or both away from each other. The times taken by light in single journeys are $\frac{r}{c-v'}$, and $\frac{r}{c+u}$ and they cannot be equal for all values of v' and u. It is this impossible assumption which is the main foundation of Relativity.

2. The above wrong assumption involves indirectly the assumption that relative velocity between two bodies can be measured somehow by a single journey of light. Now a slight reflection would make it obvious that it is impossible to conduct an experiment by sending out a messenger from one moving body to another moving body which would determine the relative velocity. If the velocity in space of the messenger is independent of the source, then he would lose touch with the body which he has left, and will be able only to measure the distance travelled by himself by noting the time taken by himself, and nothing more. Indeed it is well known that in all the experiments for measuring the velocity, light is made to perform the double journey. " The light traverses one and the same path in its journey there and back. It is only a mean velocity during the path to and fro that is actually observed." (Max Born).[14]

3. Again, if only a single journey velocity is taken, then the result would be different according as one or the other body is taken as the source, and the other the observer.

(1) A messenger sent out from B will take $\frac{r}{c+u}$ time to reach A, and then the distance between the two bodies would be

$$A'B' = r + (v'-u)\frac{r}{c+u} = \frac{c+v'}{c+u}\,r \quad . \quad . \quad . \quad (26\cdot1)$$

If A imagines himself to be at rest, then the time and distance according to him would be

$$\frac{r}{c} \text{ and } r + (v'-u)\frac{r}{c} = \frac{c+v'-u}{c}\,r \quad . \quad . \quad . \quad (26\cdot2)$$

Hence the ratio would be

$$\frac{v}{v'-u} = \frac{(c+v'-u)\,\frac{r}{c}}{\frac{r}{c}} \times \frac{\frac{r}{c+u}}{\frac{c+v'}{c+u}\cdot r}$$

$$= \frac{c+v'-u}{c+v'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26\cdot3)$$

(2) On the other hand, a messenger sent out from A will take time $\frac{r}{c-v'}$, and the distance

$$A_1B_1 = r + (v'-u)\frac{r}{c-v'} = \frac{c-u}{c-v'}r. \quad . \quad . \quad . \quad (26{\cdot}4)$$

But if B imagines himself at rest, he gets the same values for time and distance as A, *i.e.*, $\frac{r}{c}$ and $r+(v'-u)\frac{r}{c} = \frac{c+v'-u}{c}$. . . . (26·5)

Hence the ratio would be

$$\frac{v}{v'-u} = \frac{c+v'-u}{c-u} \quad . \quad . \quad . \quad . \quad . \quad (26{\cdot}6)$$

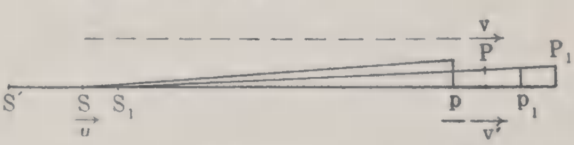
It is obvious that the values would be different in the two cases unless one wrongly assumes that $\frac{r}{c+v'} = \frac{r}{c-u}$ for all values of u and v'.

Although it has to be conceded that for a single journey the Doppler effect is different according as one body is the source or the other, Relativity astonishingly assumes that for a single journey the Relative velocity is the same.

Section II

THE METHOD OF PARALLAX

Subtended angles.



(1) Let an object of height h subtend angles α and α_1 at S and S_1, which are the positions of S at times t and t_1. Similarly, let P and P_1 be then the positions of an object P. Let the velocities of S and P be u and v' respectively with reference to S′ supposed to be at rest. Let v be the velocity of P as measured by S. The actual relative velocity of P is $(v'-u)$.

As light takes time to travel from P to S, it is obvious that the image received by S started from a nearer position p than P.

At time t, S measures the angle α subtended by the image which left p, and so $Sp = h \cot \alpha$.

But this image left p at time $\left(t - \frac{h \cot \alpha}{c}\right)$.

Again at time t_1, S_1 measures the angle α_1, which left the object from p_1 nearer than P_1.

So that $S_1\,p_1 = h \cot \alpha_1$.

But this image had left p_1 at time $\left(t_1 - \dfrac{h \cot \alpha_1}{c}\right)$.

Hence while an observer moving from S to S_1 measures the time interval as $= (t_1 - t) = \mathrm{T}$

$$\text{the real lapse of time} = \left(t_1 - \frac{h \cot \alpha_1}{c}\right) - \left(t - \frac{h \cot \alpha}{c}\right)$$

$$= \mathrm{T} - \frac{h\,(\cot \alpha_1 - \cot \alpha)}{c} = \left(\mathrm{T} - \frac{\beta}{\mathrm{C}}\right)$$

where $\beta = h\,(\cot \alpha_1 - \cot \alpha)$.

Now S measures the distance travelled by P as $= h\,(\cot \alpha_1 - \cot \alpha)$. Hence the velocity of P as measured by S is

$$v = \frac{h\,(\cot \alpha_1 - \cot \alpha)}{\mathrm{T}} = \frac{\beta}{\mathrm{T}} \quad . \quad . \quad . \quad (27{\cdot}1)$$

But the real distance travelled by P in the real time is

$$\begin{aligned} pp_1 &= SS_1 + S_1\,p_1 - Sp \\ &= \mathrm{T}u + h\,(\cot \alpha_1 - \cot \alpha) \\ &= \mathrm{T}u + \beta. \end{aligned}$$

Hence the real velocity of P is

$$(v' - u) = \frac{\mathrm{T}u + h\,(\cot \alpha_1 - \cot \alpha)}{\mathrm{T} - \dfrac{h\,(\cot \alpha_1 - \cot \alpha)}{c}}$$

$$= \frac{\mathrm{T}u + \beta}{\mathrm{T} - \dfrac{\beta}{c}} \quad . \quad . \quad . \quad . \quad . \quad (27{\cdot}2)$$

$$\therefore \quad \frac{v}{v' - u} = \frac{\beta}{\mathrm{T}} \times \frac{\left(T - \dfrac{\beta}{c}\right)}{\mathrm{T}u + \beta} = \frac{\beta}{\mathrm{T}} \times \frac{1 - \dfrac{\beta}{\mathrm{T}} \cdot \dfrac{1}{c}}{u + \dfrac{\beta}{\mathrm{T}}}$$

$$= \frac{k\left(1 - \dfrac{k}{c}\right)}{u + k} \quad . \quad . \quad . \quad . \quad (27{\cdot}3)$$

where $k = \frac{\beta}{T}$.

(2) If $u = 0$, and $\frac{\beta}{T} \cdot \frac{1}{c}$ is small compared to the relative velocities then $v = v' - u$ (Newtonian form).

In any case, as the velocities are small compared to c, and the angles α and α_1 are nearly equal, actual observations on the earth do not show any difference from the Newtonian form. Otherwise the result would be different.

Section III
THE FLASH METHOD

Let S′ be at rest and S and P move with velocities u and v'.

At time t, the positions are S and P; the previous positions of P and P_1 are p and p_1. At time t_1 the positions are S_1 and P_1.

When light flashes are received by S from P, S measures the time apparently as $(t_1 - t) = T$ . . . . (28·1)

But the flash received by S and S_1 started from p and p_1. So the real time is

$$T_0 = (t'_0 - t_0) = \left(t_1 - \frac{S_1 p_1}{c}\right) - \left(t - \frac{Sp}{c}\right)$$

$$= \left(T - \frac{S_1 p_1 - Sp}{c}\right) \quad . \quad . \quad . \quad . \quad (28·2)$$

The distance travelled by P as seen by S is $l = S_1 p_1 - Sp$, because S received messages from p and p_1 at S and S_1 . . . . . . (28·3)

But the real distance travelled by P as seen by S′ is

$$l_0 = pp_1 = Sp_1 - Sp = SS_1 + S_1 p_1 - Sp \quad . \quad . \quad . \quad (28·4)$$

Now $\qquad \frac{pP}{v'} = \frac{Sp}{c}$ and $\frac{p_1 P_1}{v'} = \frac{S_1 p_1}{c}$

But $\qquad$ time $\left(t_0 + \frac{pp_1}{v'} + \frac{S_1 p_1}{c}\right) = \left(t_0 + \frac{Sp}{c} + \frac{SS_1}{u}\right)$

and $\qquad \frac{S_1 p_1}{c} - \frac{Sp}{c} = \frac{SS_1}{u} - \frac{pp_1}{v'}$

$$= \frac{SS_1}{u} - \frac{SS_1 + S_1 p_1 - Sp}{v'}.$$

Hence $\qquad (S_1 p_1 - Sp).\left(\dfrac{1}{c} + \dfrac{1}{v'}\right) = SS_1\left(\dfrac{1}{u} - \dfrac{1}{v'}\right).$

But $\quad SS_1 = Tu.$

$$\therefore \qquad S_1 p_1 - Sp = Tu.\left(\frac{v'-u}{v'+c}\right).\frac{c}{u}.$$

Hence
$$\frac{T}{T_0} = \frac{T}{T - \dfrac{S_1 p_1 - Sp}{c}} = \frac{T}{T - T.\dfrac{v'-u}{v'+c}} \qquad . \quad . \quad (28{\cdot}5)$$

$$\therefore \quad T_0 = T\left(1 - \frac{v'-u}{v'+c}\right) = T.\frac{c+u}{c+v'} \qquad . \quad . \quad . \quad (28{\cdot}6)$$

And
$$\frac{l}{l_0} = \frac{S_1 p_1 - Sp}{S\,S_1 + (S_1 p - Sp)} = \frac{Tu.\dfrac{v'-u}{v'+c}.\dfrac{c}{u}}{Tu + Tu.\dfrac{v'-u}{v'+c}.\dfrac{c}{u}}$$

$$= \frac{\dfrac{v'-u}{v'+c}.\dfrac{c}{u}}{1 + \dfrac{v'-u}{v'+c}.\dfrac{c}{u}} = \frac{c\,(v'-u)}{v'\,(c+u)}. \qquad . \quad . \quad (28{\cdot}7)$$

The result would of course be different if *P* be the observer.

SECTION IV

THE METHOD OF WAVELENGTH OR SPECTRAL SHIFT

1. *The Doppler Effect.*

(1) Let A be the source moving with velocity u and B the observer moving with v'. Let λ and T denote the wavelength and period.

As waves come out from A, $\quad \lambda_1 = \lambda_0 - u.\,T_0 = \lambda_0 - \dfrac{u}{c}.\,\lambda_0$

$$= \lambda_0\left(1 - \frac{u}{c}\right)$$

As waves reach B′, $\qquad \lambda_2 = \lambda_1 + v'T_2 = \lambda_1 + v'\dfrac{\lambda_2}{c}$

$$\lambda_1 = \lambda_2\left(1 - \frac{v'}{c}\right)$$

Hence $$\lambda_2 = \frac{\left(1-\frac{u}{c}\right).\lambda_0}{\left(1-\frac{v'}{c}\right)} \quad . \quad . \quad . \quad . \qquad (29\cdot1)$$

$$= \left(1+\frac{v'}{c}-\frac{u}{c}-\frac{v'u}{c^2}+\frac{v'^2}{c^2}-\frac{v'^2u}{c^3}+\frac{v'^3}{c^3}\right)\lambda_0 \quad . \quad . \qquad (29\cdot2)$$

This holds good whether $v' \lessgtr u$.

(2) Let A be the observer and B the source.

As waves emerge from B, $\lambda_1 = \lambda_0 + v'.\, T_0 = \lambda_0 + v'.\frac{\lambda_0}{c} = \left(1+\frac{v'}{c}\right).\lambda_0.$

As waves reach A′, $\lambda_2 = \lambda_1 - u.\, T_2 = \lambda_1 - u\,\frac{\lambda_2}{c}$

$$\lambda_1 = \left(1+\frac{u}{c}\right).\lambda_2.$$

Hence $$\lambda_2 = \frac{\left(1+\frac{v'}{c}\right)}{\left(1+\frac{u}{c}\right)}\lambda_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29\cdot3)$$

$$= \left(1+\frac{v'}{c}-\frac{u}{c}-\frac{v'u}{c^2}+\frac{u^2}{c^2}+\frac{v'u^2}{c^3}-\frac{u^3}{c^3}\right)\lambda_0 \quad . \quad . \qquad (29\cdot4)$$

2. Now if the change in the wavelength of light were to depend exclusively on the relative velocity V of the two bodies and be independent of the absolute velocities then $\lambda + d\lambda = \left(1+\frac{V}{c}\right)\lambda$

$$\therefore \quad \frac{d\lambda}{\lambda} = \frac{V}{c}.$$

Hence if the apparent relative velocity be v and the real relative velocity $(v'-u)$, then

$$\lambda_0 + d\lambda = \left(1+\frac{v}{c}\right)\lambda_0 \quad \text{and} \quad \frac{d\lambda}{\lambda_0} = \frac{v}{c}$$

while $$\lambda_0 + d\lambda_0 = \left(1+\frac{v'-u}{c}\right)\lambda_0 \quad \text{and} \quad \frac{d\lambda_0}{\lambda_0} = \frac{v'-u}{c}.$$

According to Newton $v = v' - u$ and $\therefore\ d\lambda = d\lambda_0$.

But if $v = f.\,(v'-u)$, then $d\lambda = f.\,d\lambda_0$ . . . . (29·5)

3. Now as $V = c.\,\frac{d\lambda}{\lambda}$, if the wavelength depended on the relative velocity only, then

$$(v'-u) = c.\,\frac{d\lambda}{\lambda_0} \quad . \quad . \quad . \quad . \quad . \qquad (29\cdot6)$$

Thus the spectral shift $\frac{d\lambda_0}{\lambda_0}$ will determine the apparent relative velocity deduced on the wrong assumption.

4. It is now apparent that

(*a*) If the wavelength be wrongly supposed to depend on the relative velocity only then the apparent relative velocity v is supposed to be equal to $v'-u$ and is taken to be $c.\frac{d\lambda}{\lambda_0}$.

(*b*) But if the source be moving with velocity v' and the observer with u, then the real relative velocity

$$= (v' - u) - \frac{v'u}{c} + \frac{u^2}{c} + \frac{v'u^2}{c^2} - \frac{u^3}{c^2} \quad . \quad . \quad . \quad (29\cdot7)$$

Hence the error in the assumption

$$= - \frac{v'u}{c} + \frac{u^2}{c} + \frac{v'u^2}{c^2} - \frac{u^3}{c^2}. \quad . \quad . \quad . \quad . \quad (29\cdot8)$$

(*c*) It is also clear that the relative velocity as measured by *B* would be different and would have the error

$$= - \frac{v'u}{c} + \frac{v'^2}{c} - \frac{v'^2u}{c^2} + \frac{v'^3}{c^2} \quad . \quad . \quad . \quad . \quad (29\cdot9)$$

It follows that the relative velocity would be different according as one or the other is the observer.

SECTION V

THE DOUBLE JOURNEY METHOD

1. A new method of measuring common time and distance and also the ratio of real and apparent relative velocities based on a double journey of light was given in Ch. V, Sec. II, pp. 242—45. All experiments with light measure only the average to-and-fro velocity. Hence the double journey method is the only practical method of measuring relative velocities when a messenger, like light, is employed which has a nearly constant velocity in space. Suppose that two bodies A and B are moving with uniform velocities u and v' in space. A flash of light is sent out from a source on A and falls on a reflector on B and returns to A and the time taken for the double journey is noted as t. Then a second flash is immediately sent out

and the time taken is noted as t'. Let r and $r+\delta r$ be the distances between the bodies at the time when the flashes are sent out.

From (23·1) $t=\dfrac{2.D.r}{(D-v')(D+u)}$ and $t'=\dfrac{2.D(r+\delta r)}{(D-v')(D+u)}$

also $\dfrac{t'}{t}=\dfrac{r+\delta r}{r}$ $\therefore$ $\dfrac{t'-t}{t}=\dfrac{\delta r}{r}$.

Now the increase of the distance between the two bodies is δr in time t, *i.e.* their real relative velocity is $(v'-u)=\dfrac{\delta r}{t}=\dfrac{\delta r}{r}\cdot\dfrac{r}{t}=\dfrac{t'-t}{t}\cdot\dfrac{(D-v')(D+u)}{22D}$

$$=\frac{(D-v')(D+u)}{2D}\cdot\left(\frac{t'-t}{t}\right). \quad . \quad . \quad . \quad . \quad (30\cdot1)$$

As shown in Ch. V, Sec. II, the result is the same even if B is used as source and A as the reflector, provided the clocks at A and B keep the same absolute time.

Similarly, if A is wrongly assumed to be at rest and B moving with relative velocity $(v'-u)$ or if it be wrongly assumed with Newton that the result will be the same if A is reduced to rest by adding an equal and opposite velocity to B then from (23·5)

$$t_1=\frac{2.r'}{D-v'+u} \text{ and } t'_1=\frac{2(r+\delta r)}{D-v'+u}$$

also $\dfrac{t'_1}{t_1}=\dfrac{r+\delta r}{r}$

$$\therefore \frac{t'_1-t_1}{t_1}=\frac{\delta r}{r}$$

Hence the apparent relative velocity is

$$v=\frac{\delta r}{t_1}=\frac{\delta r}{r}\cdot\frac{r}{t_1}=\frac{t'_1-t_1}{t_1}\cdot\frac{D-v'+u}{2}$$

$$=\frac{D-v'+u}{2}\cdot\frac{t'_1-t_1}{t_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30\cdot2)$$

But as the observer and his clock are the same $\dfrac{t'_1-t_1}{t_1}=\dfrac{t'-t}{t}$

Hence $\dfrac{v}{v'-u}=\dfrac{D-v'+u}{2}\div\dfrac{(D-v')\,(D+u)}{2D}$

$$=\frac{D\,(D-v'+u)}{(D-v')(D+u)}=\frac{1-\dfrac{v'}{D}+\dfrac{u}{D}}{\left(1-\dfrac{v'}{D}\right)\left(1+\dfrac{u}{D}\right)} \quad . \quad (30\cdot3)$$

Same as already found in (23·29).

2. If a third flash is sent out immediately and is then reflected back,

then $$t'' = \frac{2D.(r+\delta r+\triangle r)}{(D-v')(D+u)} \quad \text{and} \quad t''_1 = \frac{2(r+\delta r+\triangle r)}{D-v'+u}$$

It is obvious that for each such extra observation a new unknown quantity is introduced. Hence no matter how many times the observation be repeated it will ever be impossible to know exactly the absolute velocity of either of the two bodies. In the same way it will be impossible to know the exact value of r at any point of time, though the total distance travelled by the messenger in the double journey will obviously be $D.t$.

3. It will be seen that to the first order term, the formula gives Newton's value $v=v'-u$, and to the second order term, the value

$$v = \frac{v'-u}{1-\frac{v'u}{D^2}}.$$

For higher approximations, the relative value is a function of the actual values v' and u taken separately and not only of their difference $(v'-u)$. Accordingly, the value would vary slightly with v' and u. There would also be a slight variation if the method were somewhat changed as in Ch. V, pp. 247-48.

4. Similarly, the relativity formula is true only as a second approximation, but is not rigorously true. The actual value of the apparent relative velocity depends on the particular method of observation chosen, and so is an indefinite quantity.

CHAPTER VIII

Further Corrections to Newton's Law

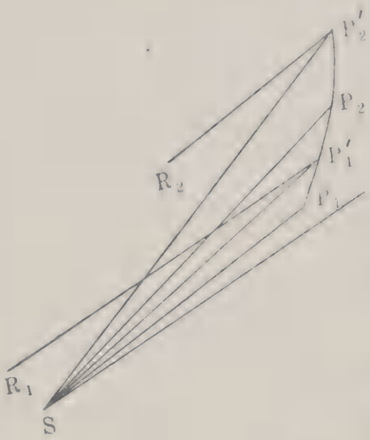
Section I

RETARDED GRAVITATION

I. *Of a stationary body.*

1. Newton assumed that Gravitational influence from the sun S would arrive at a planet P instantaneously, and would act in the same way no matter how fast and in which direction the planet may be moving. This meant that the velocity of gravitation must be infinite. If the velocity be finite then two corrections are necessary: (1) The influence when acting on the planet acts in a shifted direction, which has already been pointed

out in Chap. I, Secs. IV and V. (2) The influence will take time to arrive, with the result that the planet would have moved forward in the interval.



2. Newton assumed that the attractive pulls at P_1 along $P_1S = \frac{\mu}{r^2}$

at P_2 along $P_2S = \frac{\mu}{(r+dr)^2}$ etc. etc.

Really the pulls act at P'_1 along P'_1R_1 and at P'_2 along P'_2R_2, etc., where $SP'_1 = r+\delta r$ and $SP_2' = (r+dr)+\delta\ (r+dr)$, etc.

Hence the magnitudes of these pulls as shown in Appendix to Ch. I (pp. 259-60) are all decreased in the ratio $\frac{1}{\left(1+\frac{\delta r}{r}\right)^2} = \left(1-\frac{1}{D}\frac{dr}{dt}\right)^2$.

The frequency will add another factor $\left(1-\frac{1}{D}\frac{dr}{dt}\right)$. These results were taken into account in the equations previously.

3. Now the other effect will be that by the time the gravitational influence overtakes the planet, it will have moved further by a distance δr and rotated by an angle $\delta\theta$, where

$$\delta r = \frac{dr}{dt}\delta t = \frac{dr}{dt}\cdot\frac{r+\delta r}{D} = \frac{r}{D}\frac{dr}{dt} \text{ nearly} \quad . \quad . \quad . \quad (31\cdot 1)$$

$$\text{and } \delta\theta = \frac{d\theta}{dt}\delta t = \frac{d\theta}{dt}\frac{r+\delta r}{D} = \frac{r}{D}\frac{d\theta}{dt} \text{ nearly} \quad . \quad . \quad . \quad (31\cdot 2)$$

By substituting $(r+\delta r)$ for r and $(\theta+\delta\theta)$ for θ in (5·41) and neglecting $d\,(\delta t)$ for a nearly circular orbit we get in place of Newton's equation the following—

$$\frac{d}{dt}\left\{(r+\delta r)^2\frac{d}{dt}(\theta+\delta\theta)\right\} = \frac{\mu}{D}\frac{d}{dt}(\theta+\delta\theta) \quad . \quad . \quad . \quad (31\cdot 3)$$

$$\text{and } \frac{d^2}{dt^2}(r+\delta r)-(r+\delta r)\left\{\frac{d}{dt}(\theta+\delta\theta)\right\}^2 = -\frac{\mu}{(r+\delta r)^2}$$
$$-\frac{(m-n)\mu}{D^2}\left\{\frac{d}{dt}(\theta+\delta\theta)\right\}^2 + \frac{(m-n)}{D}\frac{\mu}{(r+\delta r)^2}\frac{d}{dt}(r+\delta r) \quad . \quad (31\cdot 4)$$

which when simplified give

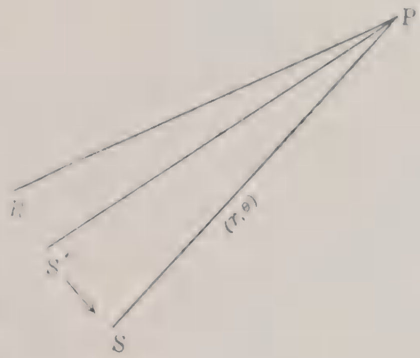
$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\mu}{D}\frac{d\theta}{dt} - \frac{2\mu}{D^2}\frac{dr}{dt}\frac{d\theta}{dt} \quad . \quad . \quad . \quad (31\cdot 5)$$

and $$\frac{d^2r}{dt^2} = -\frac{\mu}{r^2} - (m-n)\frac{\mu}{D^2}\left(\frac{d\theta}{dt}\right)^2 + (m-n+3)\frac{\mu}{D}\frac{1}{r^2}\frac{dr}{dt}$$

$$- (m-n)\frac{\mu}{D^3}\left(\frac{d\theta}{dt}\right)^2\frac{dr}{dt} - 2(m-n)\frac{\mu}{D^2}\frac{1}{r^2}\left(\frac{dr}{dt}\right)^2 \quad . \quad . \quad (31\cdot6)$$

The two extra terms on the right-hand side of (31·5) and the four extra terms on the right-hand side of (31·6) can be treated as transverse and radial disturbing forces superimposed on Newtonian force.

II. *Of a moving body.*



Let S and P be the present positions of the sun and the planet. Owing to the delay in the arrival of gravitation the influence arriving at P now really started from S′ and not S, and has come along S′P and acts along PR.

Hence the force coming along S′P $= -\frac{\mu}{(r-\delta r)^2}$ where as before $-\delta\theta = \frac{r}{D}\frac{d\theta}{dt}$

and $$-\delta r = \frac{r}{D}\frac{dr}{dt} \quad . \quad . \quad . \quad . \quad . \quad (31\cdot7)$$

The previous method can be followed in exactly the same way.

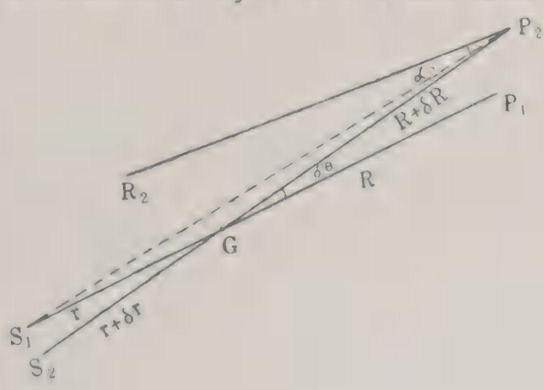
It is also apparent that in Laplace's theory the first order terms would cancel each other, and the effect on the perturbations would be practically nil.

SECTION II

TWO MOVING BODIES

Geometrical method.

1. Both the sun and the planet would obviously revolve round their common centre of gravity G in opposite directions, with equal angular velocities in such a way that their distances from G and their masses have the ratios $\frac{r}{R} = \frac{GS}{GP} = \frac{m}{M}$ and so M. SP $=(M+m)$ GP.



Hence the Newtonian force of attraction of the sun on the planet is $\frac{G.M}{SP^2}=\frac{G.M}{\left(1+\frac{m}{M}\right)^2 . GP^2}$ which is equivalent to a mass $\frac{M}{\left(1+\frac{m}{M}\right)}$ acting from G.

2. Now if S_1 and P_1 be the present positions, then instead of the gravitation coming along S_1P_1 and acting at P_1 as Newton supposed, it will come along S_1P_2 and act along P_2R_2. Also $\delta R=\frac{R+r}{D}\frac{dR}{dt}$ and

$$\delta\theta=\frac{R+r}{D}.\frac{d\theta}{dt} \qquad (32\cdot1)$$

But $d_2^2=S_1P_2^2=r^2+(R+\delta R)^2-2r(R+\delta R)\cos(\pi-\delta\theta)$

$$=R^2\left(1+\frac{m}{M}\right)^2\left[1+\frac{2}{D}\frac{dR}{dt}-\frac{m}{M}R^2\left(\frac{d\theta}{dt}\right)^2\times\left\{1+\frac{1}{D}\left(1+\frac{m}{M}\right)\frac{dR}{dt}\right\}\right] \qquad (32\cdot2)$$

$$\frac{\text{Sin }\beta_2}{\text{Sin }(\pi-\delta\theta)}=\frac{r}{d_2}$$

$$\therefore \text{Sin }\beta_2=\frac{\text{Sin }\delta\theta}{d_2}=\frac{r\delta\theta}{d_2}=\frac{r}{d_2}w.\delta t=\frac{rw}{D}=\frac{m}{MD}R\frac{d\theta}{dt} \qquad (32\cdot3)$$

$$\text{Cos }\beta_2=\sqrt{1-\frac{m^2}{M^2}\frac{w^2}{D^2}R^2}=1-\frac{1}{2}\frac{m^2}{M^2}\frac{R^2}{D^2}\left(\frac{d\theta}{dt}\right)^2$$

where β_2 = angle $S_1P_2R_2$ . . . . (32·4)

$$\text{Also Sin }\alpha_2=\frac{Rw}{D\sqrt{1+\frac{R^2w^2}{D^2}}}.\ \text{Cos }\beta_2=\frac{R}{D}\frac{d\theta}{dt}\left\{1-\frac{1}{2}\frac{R^2}{D^2}\left(\frac{d\theta}{dt}\right)^2\right\} \qquad (32\cdot5)$$

$$\text{Cos }\alpha_2=1-\frac{1}{2}\frac{R^2}{D^2}\left(\frac{d\theta}{dt}\right)^2 \text{ nearly.} \qquad (32\cdot6)$$

Hence if α be the total shift then

$\text{Sin }\alpha=\text{Sin }(\alpha_2+\beta_2)$

$$=\frac{R}{D}\frac{d\theta}{dt}\left\{1-\frac{1}{2}\frac{R^2}{D^2}\left(\frac{d\theta}{dt}\right)^2\right\}\left\{1-\frac{1}{2}\frac{m^2}{M^2}\frac{R^2}{D^2}\left(\frac{d\theta}{dt}\right)^2+\frac{m}{MD}R\frac{d\theta}{dt}\right\} \qquad (32\cdot7)$$

$$\text{Cos }a=1-\frac{1}{2}\frac{R^2}{D^2}\left(\frac{d\theta}{dt}\right)^2 \text{ nearly} \qquad (32\cdot8)$$

It is obvious that β_2 is very small as compared to α_2.

Also $\delta\theta$ is nearly $\left(1+\frac{m}{M}\right)$ times α. So that $P_2 R_2$ is nearly parallel to $S_1 P_1$. Thus the components along and normal to SP remain nearly the same; only an extra couple due to the displaced position comes into existence.

3. Now the distance between P_2R_2 and P_1S_1 is R $\delta\theta$ nearly $=\frac{R^2}{D}\frac{d\theta}{dt}\left(1+\frac{m}{M}\right)$. Hence the moment of the force whose magnitude has been shown to be practically the same as that of Newton

$$=\frac{\mu}{(R+r)^2}\cdot R.\,\delta\theta. = \frac{\mu}{R^2\left(1+\frac{m}{M}\right)^2}\cdot\frac{R^2}{D}\cdot\frac{d\theta}{dt}\left(1+\frac{m}{M}\right)=\frac{\mu}{D}\frac{d\theta}{dt}\left(1-\frac{m}{M}\right)$$

Hence $$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{\mu}{D}\frac{d\theta}{dt}\left(1-\frac{m}{M}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (32\cdot 9)$$

4. *Analytical method.*—The more approximate equations of motion can be better obtained analytically. It will be convenient to refer the motion to the centre of gravity as the fixed origin. Let the coordinates of S be (r, θ) and those of P be (R, Θ). Owing to the delay in the arrival of the gravitational influence, the planet is overtaken at P_2 (*i.e.*, $R+\delta R$ $\Theta+\delta\Theta$) instead of at P_1 (R, Θ), at time $t+\delta t$ instead of time t.

$$\text{Now } \delta R = \frac{R+r}{D}\frac{dR}{dt}=\left(1+\frac{m}{M}\right)\frac{R}{D}\frac{dR}{dt}$$

$$\delta\theta = \frac{R+r}{D}\frac{d\theta}{dt}=\left(1+\frac{m}{M}\right)\frac{R}{D}\frac{d\theta}{dt}$$

$$\frac{d}{d(t+\delta t)}=\frac{1}{\frac{d(t+\delta t)}{dt}}\frac{d}{dt}=\frac{1}{\frac{d\left(t+\frac{R+r}{D}\right)}{dt}}\frac{d}{dt}$$

$$=\frac{1}{1+\frac{1}{D}\left(1+\frac{m}{M}\right)\frac{dR}{dt}}\frac{d}{dt} \quad . \quad . \quad (32\cdot 10)$$

From equations in (5·41) and (5·7), we get the following equations of motion at $(R+\delta R)$, $(\Theta+\delta\Theta)$ at time $t+\delta t$.

$$\frac{d}{d(t+\delta t)}\left\{(R+\delta R)^2\frac{d(\theta+\delta\theta)}{d(t+\delta t)}\right\}=\frac{\mu}{D}\cdot\frac{d(\theta+\delta\theta)}{d(t+\delta t)} \quad . \quad . \quad . \quad . \quad (32\cdot 11)$$

and $$\frac{d^2(\mathrm{R}+\delta\mathrm{R})}{d(t+\delta t)^2} - (\mathrm{R}+\delta\mathrm{R})\left\{\frac{d(\theta+\delta\theta)}{d(t+\delta t)}\right\}^2 = -\frac{\mu}{(\mathrm{R}+\delta\mathrm{R})^2}$$

$$-\frac{(m-n)\,\mu}{\mathrm{D}^2}\left\{\frac{d(\theta+\delta\theta)}{d(t+\delta t)}\right\}^2 + \frac{(m-n)\mu}{\mathrm{D}}\,\frac{1}{(\mathrm{R}+\delta\mathrm{R})^2}\,\frac{d(\mathrm{R}+\delta\mathrm{R})}{d(t+\delta t)}\,. \quad (32{\cdot}11)$$

which when simplified give

$$\frac{d}{dt}\left(\mathrm{R}^2\frac{d\theta}{dt}\right) = \left\{\frac{\mu}{\mathrm{D}}\,\frac{d\theta}{dt} - 2\mathrm{R}^2\frac{d\theta}{dt}\,\frac{1}{\mathrm{D}}\left(1+\frac{m}{\mathrm{M}}\right)\frac{d^2\mathrm{R}}{dt^2}\right\}$$

$$\times\left\{1-\frac{1}{\mathrm{D}}\left(1+\frac{m}{\mathrm{M}}\right)\frac{d\mathrm{R}}{dt}\right\} \quad . \quad . \quad . \quad . \quad . \quad (32{\cdot}12)$$

and $$\frac{d^2\mathrm{R}}{dt^2} - \mathrm{R}\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{\mathrm{R}^2} - \frac{(m-n)\mu}{\mathrm{D}^2}\left(\frac{d\theta}{dt}\right)^2 + \frac{(m-n)\mu}{\mathrm{D}}\,\frac{1}{\mathrm{R}^2}\,\frac{d\mathrm{R}}{dt}$$

$$+\frac{2\mu}{\mathrm{DR}^2}\left(1+\frac{m}{\mathrm{M}}\right)\frac{d\mathrm{R}}{dt} - \frac{2(m-n)\mu}{\mathrm{D}^2\mathrm{R}^2}\left(\frac{d\mathrm{R}}{dt}\right)^2\left(1+\frac{m}{\mathrm{M}}\right)$$

$$+\frac{\mathrm{R}}{\mathrm{D}}\left(1+\frac{m}{\mathrm{M}}\right)\frac{d\mathrm{R}}{dt}\left(\frac{d\theta}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (32{\cdot}13)$$

The second method.—If S (r, θ) and P (R, Θ) be the present positions of the sun and the planet then their previous positions would be S′ $(r-\delta r, \theta-\delta\theta)$ and P′ $(\mathrm{R}-\delta\mathrm{R}, \Theta-\delta\Theta)$ at time $t-\delta t$, from which the gravitation reaches the planet now.

Then the equations of motion can be found in the same way as before by substituting $\mathrm{R}-\delta\mathrm{R}$ for R, $\Theta-\delta\Theta$ for Θ, and $t-\delta t$ for t.

Section III

REAL AND RELATIVE ORBITS

1. As the law of force is very nearly the inverse square of the distance from the common centre of gravity, it is obvious that both the sun and the planet will describe nearly elliptic orbits round the centre of gravity as focus; and each will also describe a larger ellipse relatively to the other with equal angular velocities. The major axis will depend on the magnitude of the velocity v at a distance R from G and will be given by $a = \dfrac{\mu'\mathrm{R}}{2\mu'-\mathrm{R}v^2}$ and will be independent of the direction of the velocity

at that point; while the minor axis will depend on the angle of projection β at R and will be given by

$$b = \frac{R^{3/2}}{\sqrt{2\mu^1 - Rv^2}} \cdot v \sin \beta, \text{ where } \mu^1 = \frac{G.\ M^3}{(M+m)^2}.$$

The tangential velocities of the two bodies will be in parallel and opposite directions, the bodies being on the opposite sides of G. The orbits will remain ellipses so long as $v^2 < \dfrac{2\ G.M}{\left(1+\dfrac{m}{M}\right)^2} \ \dfrac{1}{R}$.

The orbit will be a circle if $\beta = \frac{\pi}{2}$, and the value of $b = a$.

2. Now as $\frac{m}{M} = \frac{r}{R}$ and $\theta = \Theta + \pi$ it follows that

if the sun describes the ellipse $\dfrac{1}{r} = \dfrac{\mu^1}{h^2}(1 + e \cos \theta)$ . . . . (33·1)

then the planet will describe the ellipse $\dfrac{1}{R} = \dfrac{m}{M}\ \dfrac{\mu^1}{h^2}(1 - e \cos \theta)$. (33·2)

and both the sun and the planet will describe relatively round each other the ellipse $\dfrac{1}{\rho} = \dfrac{1}{(R+r)} = \dfrac{m}{M+m}\ \dfrac{\mu^1}{h^2}(1 + e \cos \theta)$ . . . . (33·3)

3. More generally, if the law of attraction be $\dfrac{G.M.m}{(R+r)^2}$, then the force will be $\dfrac{G.\ M.\ m}{R^2\left(1+\dfrac{m}{M}\right)}$ . . . . (33·4)

If the path of the planet round the centre of gravity be $R = F(\theta)$ the path of P relative to S will obviously be

$$\rho = (R+r) = \left(1 + \frac{m}{M}\right) F(\theta) \quad . \quad . \quad . \quad . \quad . \quad (33·5)$$

4. If the orbits be nearly circular then approximately R and r are constants and $\frac{dR}{dt}$ and $\frac{dr}{dt}$ are both negligible, and $\frac{d\theta}{dt} = w$ a constant. It is also clear that each planet will have the same angular velocity round the other as both have round the centre of gravity. It is also clear that the radial velocity v is zero and the transverse velocities are constants Rw and rw.

Accordingly the equations of motion are

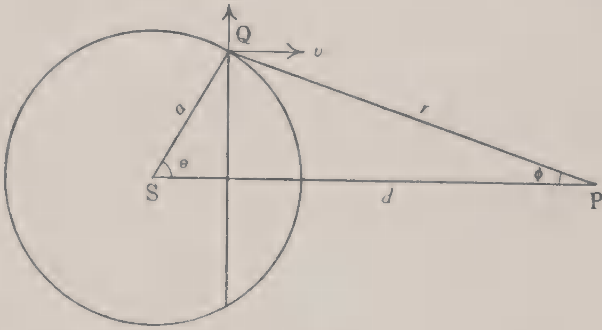
$$\frac{d}{dt}\left(R^2\frac{d\theta}{dt}\right)=\frac{\mu}{D}\frac{d\theta}{dt}=\frac{\mu}{D}w. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad 33\cdot6)$$

$$\text{and } -R\left(\frac{d\theta}{dt}\right)^2=-\frac{\mu}{r^2}-\frac{3\mu}{D^2}\left(\frac{d\theta}{dt}\right)^2, \; i.e., +Rw^2=+\frac{\mu}{r^2}+\frac{3\mu}{D^2}w^2. \qquad (33\cdot7)$$

SECTION IV

LONGITUDINAL, TRANSVERSE AND ROTATIONAL MASSES

1. Newton assumed that the attraction of a spherical shell on a point is as if the whole mass were concentrated at the centre, whether the body is moving or is stationary. Really for a moving body a slightly different mass is concentrated at the centre. Let a spherical shell with centre S have the radius a and surface density σ; and let a point Q on



the surface be such that the angle QSP $=\theta$ and QPS $=\phi$, where P is the attracted point at a distance d from S. Then the force exerted along SP by a thin annulus at Q is $f=\dfrac{\sigma 2\pi\ a\sin\theta.\ ad\theta}{r^2}\cos\phi$.

But $r^2=a^2+d^2-2ad\cos\theta$. $\quad\therefore\ rdr=ad\sin\theta\ d\theta$.

Let v be the velocity of S towards P. Then the component velocity of Q along QP$=v\cos\phi$. Hence the factor due to the Doppler and aberration effects $=\left(1+\dfrac{v\cos\phi}{D}\right)^{m-n}$

The components perpendicular to SP cancel each other; and so do the moments caused on P by the transverse components at corresponding points.

Accordingly the total force along SP is

$$F_1=-\int\frac{\sigma\,2\,\pi a^2\sin\theta d\theta}{r^2}.\quad\cos\phi\left(1+\frac{v}{D}\cos\phi\right)^{m-n}$$

$$=-\frac{\sigma\pi a}{d^2}\int\frac{r^2+(d^2-a^2)}{r^2}\left\{1+(m-n)\frac{v}{D}.\frac{r^2+(d^2-a^2)}{2rd}\right\}dr,\text{ nearly.}$$

$$= -\frac{\sigma\pi a^2}{d^2}\int \left\{ \left(1+\frac{d^2-a^2}{r^2}\right) + (m-n)\frac{vr}{2\mathrm{D}d} \right\} dr, \quad \text{nearly}$$

$$= -\frac{\sigma 4\pi a^2}{d^2}\left[1+\frac{(m-n)}{4}\frac{v}{\mathrm{D}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34{\cdot}1)$$

Thus it slightly differs from Newton's value $\frac{\sigma 4\pi a^2}{d^2}$. It follows that although the mass of the moving body is not changed, its effective attraction is increased as if the mass were increased. This can be wrongly supposed to be an increased *longitudinal* mass.

2. If u be the velocity of the spherical shell at right angles to SP then similarly the velocities of Q and Q′ along PQ and PQ′ are $\pm u \sin\phi$. Hence the total force of attraction along SP

$$= -\int \frac{\sigma\, 2\pi\, a^2 \sin\theta\, d\theta}{r^2}\cos\phi \left\{ \left(1-\frac{u}{\mathrm{D}}\sin\phi\right)^{m-n} + \left(1+\frac{u}{\mathrm{D}}\sin\phi\right)^{m-n} \right\}$$

$= -\frac{\sigma\, 4\pi\, a^2}{d^2}$, the same as under Newton's Law if $\frac{u^2}{\mathrm{D}^2}$ be neglected and ϕ be small . . . . (34·2)

The components perpendicular to SP will in addition give

$$-\int \frac{\sigma\, 2\pi a^2 \sin\theta\, d\theta}{r^2}\sin\phi\left(\frac{2(m-n)}{\mathrm{D}}\sin\phi\right) \quad . \quad . \quad . \quad (34{\cdot}3)$$

which will obviously be small if d is large and therefore ϕ is small. This can be wrongly supposed to be a changed *transverse* mass.

3. Suppose the spherical shell is merely rotating with angular velocity w, parallel to the plane containing SP, then the points Q and Q′ have velocities aw with components $\pm\, aw \sin\theta$ parallel to SP and $+\, aw\cos\theta$ perpendicular to SP. The velocity $aw\cos\theta$ is similar to u in §2 with similar results. But the components parallel to SP being equal and opposite will almost cancel each other so far as the translational force is concerned. The attraction of Q being less than that of Q′, there would be a net backward attraction at P normal to SP. Corresponding discs above and below the plane of the paper will produce similar results, their effect normal to the plane of the paper being balanced. This can be wrongly supposed to be a changed *rotational* mass.

4. If P be a body of finite dimensions instead of a particle, then at each point the net backward attraction would be:—

$$-\frac{\mu}{r^2}\left[\left\{1-\frac{aw\sin(\theta+\phi)}{\mathrm{D}}\right\}^{m-n}\sin\phi - \left\{1+\frac{aw\sin(\theta+\phi)}{\mathrm{D}}\right\}\sin\phi\right]$$

$$= -\frac{\mu}{r^2}\left[-\frac{(m-n)\,aw\sin(\theta+\phi)}{\mathrm{D}}\right]\sin\phi \quad . \quad . \quad . \quad . \quad . \quad (34{\cdot}4)$$

Hence points of P which are nearer to S will have greater backward attraction than those more distant. Hence the body P in addition to the backward translational motion will experience an angular motion in the same sense as the rotation of S.

5. The more complicated problem when both the bodies are moving and are of comparable dimensions is postponed to a later chapter.

Section V

THE MOTION OF THE SOLAR SYSTEM

As is well known the solar system as a whole is moving through space, say with velocity ω. The velocity of gravitation in free space being constant (which is like the assumption for light in Relativity), another correction is necessary. If β be the angle which the major axis of the orbit of a planet makes with the direction of the translational motion of the solar system, then $\theta+\beta$ is the angle which the radius vector makes with that direction.

Hence the velocity along the radius vector is $\dfrac{dr}{dt}+\omega.\cos(\theta+\beta)$

and that along the transversal is $\dfrac{rd\theta}{dt}-\omega.\sin(\theta+\beta)$.

Therefore the equation of motion would become

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{\mu}{D}\cdot\frac{d\theta}{dt}\text{ nearly} \quad . \quad . \quad . \quad . \quad (35{\cdot}1)$$

$$\text{and } \frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-\frac{\mu}{r^2}\Bigg[1-\frac{1}{D}\Bigg\{\frac{dr}{dt}-\frac{1}{D}\left(r\frac{d\theta}{dt}\right)^2+$$

$$\omega\left(\cos\overline{\theta+\beta}+\frac{1}{D}\frac{rd\theta}{dt}\sin\overline{\theta+\beta}\right)\Bigg\}\Bigg]^{m-n}$$

$$=-\frac{\mu}{r^2}-\frac{(m-n)}{D^2}\cdot\frac{h^2(1+k\theta)^2}{r^4}$$

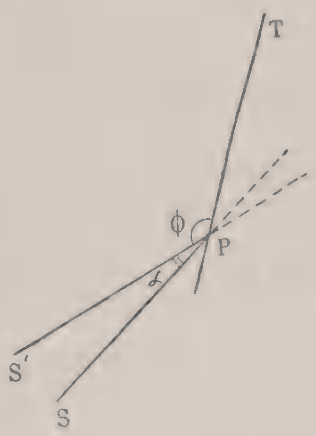
$$+\frac{(m-n)}{D}\cdot\frac{1}{r^2}\left[\frac{dr}{dt}+\omega\left\{\cos\overline{\theta+\beta}+\frac{h(1+k\theta)}{D}\cdot\frac{1}{r}\sin\overline{\theta+\beta}\right\}\right] \quad . \quad (35{\cdot}2)$$

Section VI

ELONGATED ORBITS OF COMETS

1. The equations in Ch. I, Sec. V, pp. 8—10, were true for a heavenly body moving in a nearly circular orbit for which $\frac{v'}{D}$ was small so that $\sin\alpha = \frac{1}{D}\,\frac{rd\theta}{dt}$ and $\cos\alpha = 1$. The magnitude of the force also was taken to be nearly the same along the effective direction. For a comet although the tangential velocity v' as compared to D is still small, the orbit is not at all circular. For such an orbit, as shown in Ch. V, Sec. IV, p. 250, the relations are

$$\frac{\sin\alpha}{v'} = \frac{\sin(\phi-\alpha)}{D} = \frac{\sin(\pi-\phi)}{\sqrt{D^2+2v'D\cos\phi+v'^2}}$$



where ϕ is the angle between the radius vector and the tangent. Obviously

$$\frac{dr}{dt} = v'\cos(\pi-\phi) = -v'\cos\phi$$

and $$\frac{rd\theta}{dt} = v'\sin(\pi-\phi) = v'\sin\phi$$

Accordingly $$\sin\alpha = \frac{\dfrac{rd\theta}{dt}}{\sqrt{D^2+\left(\dfrac{dr}{dt}\right)^2+\left(r\dfrac{d\theta}{dt}\right)^2-2D\dfrac{dr}{dt}}}$$

$$= \frac{1}{D}\,\frac{rd\theta}{dt}\left(1+\frac{1}{D}\,\frac{dr}{dt}\right) \text{ roughly} \quad . \quad . \quad . \quad (36{\cdot}1)$$

and $$\cos\alpha = 1-\frac{1}{2}\,\frac{r^2}{D^2}\left(\frac{d\theta}{dt}\right)^2-\frac{1}{D^2}\left(\frac{dr}{dt}\right)^2 = 1 \text{ roughly} \quad . \quad (36{\cdot}2)$$

2. On the assumption, as before, that the magnitude remains nearly the same, the equations of motion become

$$\frac{1}{r}\,\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\mu}{r^2}\left[1+\frac{v'}{D}\cos(\phi-\alpha)\right]^{m-n}\cdot\frac{v'}{D}\sin(\phi-\alpha)$$

$$= \frac{1}{r}\cdot\frac{\mu}{D}\cdot\frac{d\theta}{dt} \text{ only roughly.}$$

and $$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-\frac{\mu}{r^2}\left[1+\frac{v}{D}\cos(\phi-\alpha)\right]^{m-n} \quad \text{nearly}$$

$$=-\frac{\mu}{r^2}\left[1+\frac{(m-n)}{D^2}\left(r\frac{d\theta}{dt}\right)^2-\frac{(m-n)}{D}\cdot\frac{dr}{dt}\right]$$

only very roughly.

The rough approximations can be true for nearly circular orbits only.

SECTION VII
LIGHT RADIONS

1. It was shown in Ch. II, Sec. II, pp. 25-26, that on the assumption that the equation

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}+\frac{3\mu}{D^2}\cdot u^2$$

would to a fairly close approximation hold for light, the maximum value of $h=r.\,c$, and therefore the maximum value of the deflection of light would be $\frac{8}{3}$ times that of Newton's $=2''\cdot32$ nearly, if $D=c$.

2. Now on the same assumption the maximum value of the deflection of light can be easily calculated.

Following the method of Ch. I, Sec. VIII, pp. 12—13, and taking R as the shortest distance from the Sun, the first solution is $u=\frac{\cos\theta}{R}$.

And therefore $$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}+\frac{3\mu}{c^2R^2}\cdot\cos^2\theta.$$

It will be seen that $u_1=\frac{\mu}{h^2}+\frac{\mu}{c^2R^2}(\cos^2\theta+2\sin^2\theta)$ is a particular integral.

So that $$u=\frac{\mu}{h^2}+\frac{\cos\theta}{R}+\frac{\mu}{c^2R^2}(\cos^2\theta+2\sin^2\theta)$$

i.e., $$R=\frac{\mu}{h^2}R.r+r\cos\theta+\frac{\mu}{c^2R}(r\cos^2\theta+2r\sin^2\theta)$$

$$=\frac{\mu}{h^2}R\sqrt{x^2+y^2}+x+\frac{\mu}{c^2R}\cdot\frac{(x^2+2y^2)}{\sqrt{x^2+y^2}}$$

$\therefore$ $$x=R-\frac{\mu}{h^2}R\sqrt{x^2+y^2}-\frac{\mu}{c^2R}\cdot\frac{(x^2+2y^2)}{\sqrt{x^2+y^2}}$$

The asymptotes are found by taking y very large compared with x, then $$x = \mathrm{R} - \frac{\mu}{h^2}.\ \mathrm{R}(\pm y) - \frac{\mu}{c^2\mathrm{R}}(\pm 2y)$$

Obviously the minimum value of $h = c.\mathrm{R}$. Hence the maximum deflection of light given by the angle between the two asymptotes $= \frac{6\mu}{c^2\mathrm{R}}$, *i.e.*, three times the Newtonian value $= 2''\cdot 61$ nearly.

Thus on this method the real value may, speaking roughly, be somewhere round about $2''\cdot 46$.

3. (1) For radions $1 = \frac{c}{\mathrm{D}} = \frac{\sin\alpha}{\sin(\phi-\alpha)}$, and so $\alpha = \frac{\phi}{2}$.

Also $-c\cos\phi = \frac{dr}{dt}$, and $c\sin\phi = r\frac{d\theta}{dt}$.

The apparent velocity of gravitation along the shifted direction $= \sqrt{\mathrm{D}^2 + c^2 - 2\mathrm{D}c\cos(\pi - \phi)} = \mathrm{D}.2\cos\frac{\phi}{2}$.

(2) On the assumption that the method of Ch. I nearly holds good for light also, the equations of motion become

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\mu}{r^2}\left[1 + \left\{\cos\phi\cos\frac{\phi}{2} + \sin\phi\sin\frac{\phi}{2}\right\}\right]^{m-n}\sin\frac{\phi}{2}$$

$$= \frac{\mu}{r^2}\left(1 + \cos\frac{\phi}{2}\right)^{m-n}\sin\frac{\phi}{2} \quad . \quad . \quad . \quad . \quad . \quad (37\cdot 1)$$

and $$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2}\left[1 + \cos\frac{\phi}{2}\right]^{m-n}\cdot\cos\frac{\phi}{2} \quad . \quad . \quad . \quad . \quad . \quad (37\cdot 2)$$

(3) If ε be the small angle between the directrix and the tangent, then $\phi = \theta + \varepsilon$.

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\mu}{r^2}\left(1 + \cos\frac{\theta}{2} - \frac{\varepsilon}{2}\sin\frac{\theta}{2}\right)^{m-n} \times \left(\sin\frac{\theta}{2} + \frac{\varepsilon}{2}\cos\frac{\theta}{2}\right)$$

and $$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2}\left(1 + \cos\frac{\theta}{2} - \frac{\varepsilon}{2}\sin\frac{\theta}{2}\right)^{m-n}\left(\cos\frac{\theta}{2} - \frac{\varepsilon}{2}\cdot\sin\frac{\theta}{2}\right).$$

(4) As when light is receding from a body and gravitation cannot overtake it, the Doppler effect is not produced in the ordinary way, an additional correction will be introduced in a later chapter.

4. On the assumption that the Doppler principle and the Aberration principle are effective so long as light is approaching the sun, but cease to be effective when light begins to recede, and therefore gravitation cannot overtake it, the value of the deflection of light from a star past the sun and reaching the earth can be reduced nearly to

$$\varepsilon = -\frac{\mu}{c^2 R}\left[\int_0^{\frac{\pi}{2}}\left(1+\cos\frac{\phi}{2}\right)^3 \sin\frac{\phi}{2}\,d\phi + \int_{\frac{\pi}{2}}^{\pi} \sin\phi\,d\phi\right]$$

$$= -4{\cdot}76.\ \frac{\mu}{c^2 R} \quad \text{nearly}$$

i.e., 2·39 times the Newtonian value

$= 2''{\cdot}08$ nearly.

5. The dynamics of a changing force moving with a finite velocity and acting on a moving body is complex. Care has to be taken in compounding them, because the force causes acceleration which is of course a quantity of a different dimension from velocity. If the force were uniform or were propagated like plane waves, the simple Doppler principle would apply. But if the force is propagated like spherical waves diminishing with the distance, then a new factor appears as shown in the Appendix to Ch. I, pp. 259-60. And if the body were moving not only radially, but also possess a transverse velocity, the principle of aberration has also to be applied. Whether the magnitude of the effective force along the resultant is not changed as well will also have to be considered. In a later chapter, the problem will be re-examined from a slightly different standpoint, and the equations of motion themselves reconsidered.

CHAPTER IX

The Resisting Medium

SECTION I

THE GENERAL CASE

1. There is a growing evidence that interstellar space is filled with rarefied matter, and there is greater evidence that there are clouds of matter within the Solar System through which planets pass. The reason why in the last century a resisting medium was not accepted was that it did not

seem to have any apparent effect on the motion of planets. But the existence of Gegenschein, the shortening of the period of Enke's comet, calcium absorption and the space reddening of light are now recognised as the result of such a medium. It is far more unreasonable to suppose that space is a complete void than that some thin matter is floating about in it.

Now the resisting medium must be composed of two parts superimposed on each other (1) the interstellar matter and (2) the solar matter. The density of the former would obviously be independent of the distance of the planet from the sun, while that due to the latter would be proportional to the density, some function of r say $\rho=\psi(r)$. It is natural to suppose that the resistance due to the solar medium would decrease and not increase with the distance. It is equally clear that the acceleration R denoting the force of resistance would increase with the velocity of the planet and would be some function $F(v)$ of the velocity. It must also be proportional to the section $\pi\alpha^2$ of the planet. The acceleration would also be inversely proportional to the mass m of the planet. Again the shape of the orbit may as well have some effect and the constant may be some function $f(a, e, w)=K$ different for different planets.

Thus the generalised form of the acceleration due to a resisting medium can be put down as

$$R=-f(a, e, w).\ F(v).\ \psi(r)\ \frac{\pi\alpha^2}{m}$$

where $f(a, e, w)$ would approximately be constant for each orbit, particularly in major planets whose orbits are nearly circular;

$$F(v)=A_0+A_1v+A_2v^2+A_3v^3+\ \dots$$

where A's are constants

$$\text{and } \psi(r)=B_0+\frac{B_1}{r}+\frac{B_2}{r^2}+\frac{B_3}{r^3}+\ \dots$$

where B's are constants.

But if the solar medium is concentrated in the form of concentric shells as is quite natural, it would not be a simple function.

The General Case of Resisting Medium has been treated at length by F. Tisserand.[14]

Let $R=K.\ F(v)\ \psi.(r)$

where $\rho=\psi(r)$ and $K=c.\dfrac{\pi\alpha^2}{m}$.

Then $$R=-K\sum\left(A_nv^n\right)\cdot\sum\left(\frac{Bm}{r^m}\right) \qquad (38{\cdot}1)$$

2. As shown in Besant's Dynamics[15], for a small tangential force, we have from $v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \frac{\mu(2a-r)}{r.\,a}$

$$2\,vdv = +\frac{\mu}{a^2}\,da.$$

$$\frac{da}{dt} = \frac{2a^2 v}{\mu}\cdot\ \text{R, where the acceleration R} = \frac{dv}{dt} \quad \ldots \quad (38\cdot2)$$

$$\text{Also}\ \frac{dw}{dt} = \frac{2\,r.va.\text{R}\sin\theta}{\mu e(2a-r)} \quad \ldots \quad (38\cdot3)$$

$$\text{and}\ \frac{de}{dt} = 2.\ \text{R}.\ (\cos\theta + e)\sqrt{\frac{a.\,r}{\mu(2a-r)}} \quad \ldots \quad (38\cdot4)$$

Substituting the value of r from $\frac{a(1-e^2)}{r} = 1 + e\cos\theta$,

$$\text{we get } v = \sqrt{\frac{2\mu(1+e\cos\theta)}{a(1-e^2)} - \frac{\mu}{a}} = \sqrt{\frac{\mu(1+e^2+2e\cos\theta)}{a(1-e^2)}}$$

$$\text{Hence}\ \frac{da}{dt} = \frac{2.a^2\sqrt{\mu}.\ \sqrt{1+e^2+2\ \cos\theta}}{\mu\ \sqrt{a(1-e^2)}}\cdot\text{R}$$

$$= \frac{2\text{R}}{n\sqrt{1-e^2}}\cdot\ \sqrt{1+e^2+2e\cos\theta}. \quad \ldots \quad (38\cdot5)$$

$$\frac{de}{dt} = 2\text{R}.\ (\cos\theta + e)\sqrt{\frac{a.\,(1-e^2)}{\mu(1+e^2)+2e\cos\theta)}}$$

$$= \frac{2\text{R}\sqrt{1-e^2}.\ (\cos\theta + e)}{n.\,a.\sqrt{1+e^2+2e\cos\theta}} \quad \ldots \quad (38\cdot6)$$

$$e\,\frac{dw}{dt} = \frac{2\text{R}.\ \sin\theta.\ \sqrt{a(1-e^2)}}{\sqrt{\mu}\ (1+e^2+2e\cos\theta)}$$

$$= \frac{2\text{R}\sqrt{1-e^2}}{n.\,a}\ \frac{\sin\theta}{\sqrt{1+e^2+2e\ \cos\theta}} \quad \ldots \quad (38\cdot7)$$

For a resisting force, we have to put $-\text{R}$ for R in the above equations. These formulæ were given by Tisserand.

3. It follows that for a resisting medium, $\frac{da}{dt}$ is always negative and varies with the anomaly θ. And so a diminishes incessantly after each revolution; n increases proportionately to the number of the revolutions. There is a secular diminution with parameter $a(1-e^2)$. But the eccentricity depends on the factor $\cos\theta + e$ and so $\frac{de}{dt}$ is sometimes positive and sometimes negative.

4. Each term of the product of the summations in §1 is of the form

$$-\mathrm{K}\,\frac{v^p}{r^q}$$ (See Routh's Dynamics, Art. 384, p. 247.)

For this we get

$$\frac{1}{a}\,\frac{da}{dt} = \frac{-2\mathrm{K}}{1-e^2}\,(1+e^2+2e\cos\theta).\,\frac{v^{p-1}}{r^q} \quad . \quad . \quad . \quad (38{\cdot}8)$$

$$\frac{de}{dt} = -\,2\mathrm{K}.\ (\cos\theta+e)\,\frac{v^{p-1}}{r^q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38{\cdot}9)$$

$$e\,\frac{d\omega}{dt} = -\,2\mathrm{K}.\ \sin\theta.\,\frac{v^{p-1}}{r^q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (38{\cdot}10)$$

If we substitute $dt = \frac{r^2 d\theta}{\sqrt{\mu a(1-e^2)}}$, we have

$$\frac{1}{a}\,\frac{da}{d\theta} = \frac{-2\mathrm{K}'}{1-e^2}\,(1+e^2+2\,e\cos\theta)^{\frac{p+1}{2}}.\ (1+e\cos\theta)^{q-2} \quad . \quad (38{\cdot}11)$$

$$\frac{de}{d\theta} = -2\mathrm{K}'\,(\cos\theta+e)\,(1+e^2+2e\cos\theta)^{\frac{p-1}{2}}.\ (1+e\cos\theta)^{q-2} \quad . \quad (38{\cdot}12)$$

$$e\,\frac{d\omega}{d\theta} = -\,2\mathrm{K}'\sin\theta\,(1+e^2+2e\cos\theta)^{\frac{p-1}{2}}\,(1+e\cos\theta)^{q-2} \quad . \quad (38{\cdot}13)$$

$$\mathrm{K}' = \mathrm{K}\,\frac{\mu^{\frac{p-2}{2}}}{\{a(1-e^2)\}^{\frac{p}{2}+q-2}} = \mathrm{K}\,n^{p-2}\,a^{p-1}(1-e^2)^{2-\frac{p}{2}-q}$$

where $\mu = n^3a^2$ . . . . (38·14)

Section II

A PARTICULAR CASE

1. For a particular planet K can be treated as a constant.

The density ρ may be regarded as a constant B_0 due to the inter-stellar matter + a function of r, the distance from the sun.

Now the resisting medium due to emanations from the sun as well as matter attracted by the sun would naturally have $\rho = \frac{B_2}{r^2}$, which may be assumed as approximately true.

$$\text{Hence} \qquad f(a, e, w) = K$$

$$\text{and} \qquad \psi(r) = B_0 + \frac{B_2}{r^2}$$

where obviously B_0 should be extremely small as compared to B_2 and may in the first instance be neglected.

As regards $F(v)$, it must naturally increase with v. For small velocities (up to sound) of material bodies passing through a dense medium like air, it was found by Bashforth[16] that if v be the velocity, measured in feet per second, d the diameter of the leadshot in inches, w the weight in pounds, then taking the resistance to be $\beta \frac{d^2}{w}\left(\frac{v}{1000}\right)^n$

$v < 850$	$n=2$	$\beta=61{\cdot}3$
$v > 850 < 1040$	$n=3$	$\beta=74{\cdot}4$
$v > 1040 < 1100$	$n=6$	$\beta=79{\cdot}2$
$v > 1100 < 1300$	$n=3$	$\beta=108{\cdot}8$
$v > 1300 < 2700$	$n=2$	$\beta=141{\cdot}5$

It may be a fair assumption to make that if the density of the medium decreases, and the velocity increases to several miles per second, $n = 1$ nearly.

Accordingly as a first approximation the law of resistance applicable to planets may be taken to be

$$R = -K.\frac{v}{r^2} \qquad . \quad . \quad . \quad . \quad (38{\cdot}15)$$

2. If $R = -K.\frac{v}{r^2}$, then $p=1$ and $q=2$.

Hence (1) $\frac{1}{a}\frac{da}{d\theta} = \frac{-2K^1}{1-e^2}(1+e^2+2e\cos\theta)$ where $K^1 = \frac{K}{\sqrt{\mu a(1-e^2)}}$.

$$\frac{\Delta a}{a} = -\frac{2K^1}{1-e^2}\Big[(1+e^2)\,\theta + 2e\sin\theta\Big]_0^{2\pi}$$

$$= \frac{-2K}{\sqrt{\mu a}(1-e^2)^{3/2}} 2\pi\ (1+e^2)$$

$$= \frac{2k\mu a \ .\ 2\pi\ (1+e^2)}{h^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38\cdot16)$$

(2) $\dfrac{de}{d\theta} = -2K^1 (\cos\theta + e)$ where $K^1 = \dfrac{K}{\sqrt{\mu a(1-e^2)}}$

$$\therefore \Delta e = -2K^1 \left[\sin\theta + e.\theta\right]_0^{2\pi}$$

$$= -\frac{4\pi e.\ K}{\sqrt{\mu a(1-e^2)}} = -\frac{4\pi e}{h}.\ K \quad . \quad . \quad . \quad (38\cdot17)$$

(3) $e\dfrac{d\omega}{d\theta} = -2K^1 \sin\theta$

$$e \triangle \omega = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38\cdot18)$$

3. Generally, when the resistance is a function of v and r only it has no permanent effect on the longitude of the perihelion, but decreases both the semi-major axis and their eccentricity. The fact is that in such a case the tangential retardation causes the line of apsides to rotate backward during the one-half of the revolution and to rotate forward during the other half, the net result being nil.

The resisting medium has of course no component along the normal. But there is a normal component derived from the resultant force along the shifted direction. The normal component cannot affect the velocity, hence the semi-major axis is unchanged. A normal component rotates the line of apsides in one direction and then in the other direction of the two sections made by the latus rectum. Similarly, a normal force decreases the eccentricity during one-half of the revolution and increases it during the other half.

4. Now the tangential acceleration $-\ K.\dfrac{v}{r^2}$ can be resolved into two components one along the radius vector $= -K.\dfrac{v}{r^2}.\dfrac{\frac{dr}{dt}}{v} = -\dfrac{K}{r^2}\dfrac{dr}{dt}$. (38·19)

and the other along the transversal $= -K.\dfrac{v}{r^2}.\dfrac{\frac{rd\theta}{dt}}{v} = -\dfrac{K}{r}\dfrac{d\theta}{dt}$. (38·20)

Comparing these components with the additional terms in the equations of motion

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} - \frac{3\mu}{D^2}\frac{h^2}{r^4} + \frac{3\mu}{D}\frac{1}{r^2}\frac{dr}{dt}$$

and
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\mu}{D}\cdot\frac{1}{r}\frac{d\theta}{dt}\left(1 - \frac{2}{D}\frac{dr}{d}\right)$$

it is apparent that the retardation caused by the resisting medium and the automatic acceleration can nearly balance each other if K is of the order $\frac{\mu}{D}$, which is natural as μ is the gravitational mass of the sun from which emanations proceed and D is the velocity at which emanations travel and maintain the density. The effect of the two additional terms on the eccentricity from Ch. 1. Sec. XII is

$$\triangle e = 6\pi e \frac{\mu}{Dh}$$

whereas the change in eccentricity due to the resisting medium as found in §2 (2) of this section is $\triangle e = -\frac{4\pi e}{h}K$ . . . . (38·21)

These will almost cancel each other if $\frac{3\mu}{2D} = K$ nearly.

The perturbations in the major axis will be similarly reduced. Thus the resistance can as a last resource fully counteract the automatic acceleration, reducing the observed changes in these elements if the radionic density be of the order 10^{-12}.

5. If the law of resistance involve a higher power of v, say v^2, then the density of the radionic medium near Mercury's orbit would be very considerably reduced, and be of the order 10^{-18}; and if it were of v^3, then the density would tally with the known density of the inter-stellar space 10^{-24}.

6. Now if the tangential resistance were not only a function of v and r but say $= -\frac{k}{r^2}(v+dv)^n$ where dv represents the increase in velocity along the tangent due to the shifted attractive force, then the transverse and radial components of the tangential resistance would be $-\frac{k}{r^2}(v^{n-1}+ndv)\,r\frac{d\theta}{dt}$ and $-\frac{k}{r^2}(v^{n-1}+ndv)\frac{dr}{dt}$ nearly. With appropriate values of k and n, the effect of the extra terms in the attractive forces which cause perturbations in the major axis, the eccentricity and even the longitude of the perihelion can be substantially reduced, and even made to vanish. This can certainly be the case if the law of resistance be a more complex function $f(v, dv)$, which will be considered later.

Section III

THE OBSERVED PERTURBATIONS

After taking immense pains for a long number of years, Newcomb (1895) calculated the values of the perturbations of the Four Minor Planets. He found out the observed values from calculation and also calculated the values theoretically according to Newton's law after taking into account the effect of the other planets also. The following table[17] gives his results so far as the changes in the eccentricity per century are concerned.

	Observation	Theory	Difference		
			Minimum	Maximum	Mean
Mercury	$+3''{\cdot}36$	$+4''{\cdot}24$	$-0''{\cdot}38$	$-1''{\cdot}38$	$-0''{\cdot}88$
Venus	$-9''{\cdot}46$	$-9''{\cdot}67$	$-0''{\cdot}10$	$+0''{\cdot}52$	$+0''{\cdot}21$
Earth	$-8''{\cdot}55$	$-8''{\cdot}57$	$-0''{\cdot}08$	$+0''{\cdot}12$	$+0''{\cdot}02$
Mars	$+19''{\cdot}00$	$+18''{\cdot}71$	$+0''{\cdot}02$	$+0''{\cdot}56$	$+0''{\cdot}29$

Taking the mean values of the differences as the basis for consideration, it follows that in the case of Mars, Earth and Venus, the observed eccentricity is in excess of the theoretical one, while in the case of Mercury, it is deficient. The observations cannot be regarded as exact, but they can be accepted as tolerably accurate so far as the signs of the changes are concerned.

Now no existing theory, whether of Newton, Laplace or Einstein can explain an increase in eccentricity in a planetary orbit howsoever small Newcomb's observations of the orbits of Mars, Earth and Venus show that there are substantial increases. The known theories utterly fail to explain them. Newton's and Einstein's theories would give no increases, whereas Laplace's theory would give large decreases. These theories also fail to explain a change of sign for Mercury.

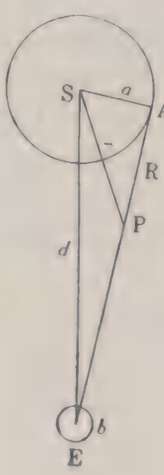
But the New Relativity Theory is the only theory which shows that there ought to be increases in the eccentricity. No doubt the increase predicted by it *prima facie* gives an increase of the order of $\frac{\mu}{h_0} = k = 10^{-4}$, which is rather large. The explanation of the resisting medium given

in Sec. II can meet this objection. For more distant planets, the resistance is small, and so the automatic acceleration still exceeds the retardation caused by the medium. But as Mercury is closer to the Sun, the resistance is greater, and the decrease in eccentricity due to the resistance is in excess of the increase due to the shifted direction, and the sign of the discrepancy in the case of Mercury is changed from positive to negative. This effect is observed still more markedly in the case of Enke's comet, which passes closer to the Sun, and its period is found to be appreciably reduced after each revolution

Appendix to Chapter II, Section III

1. It will be shown in Chapter XI that there is a great uncertainty as to the extent of the spectral shift from the sun. The displacement is greater or less according as the level is higher or lower. The supposed agreement with the Relativity values for the centre and the edge is only for arbitrarily chosen levels, and these levels differ for the two. It seems that light loses in frequency as it is emitted from a higher level. Obviously light from the centre comes from a deeper level, the visible interior, than that from the edge, the visible exterior. The observed value for the centre is unreliable because of probable convection currents, and that from the edge because of its level. For the true values, the real test will be furnished by (1) light from the highest levels only, and (2) by taking the average of $M\,\Sigma\frac{a\,d\lambda}{\lambda}$ and not $Ma\,\frac{\Sigma d\lambda}{\Sigma\lambda}$.

2. If the path of light from an edge of the sun be considered to be straight, then following the method in Ch. II, Sec. III, pp. 27-28, the spectral shift can be calculated as follows:—



As the component of the velocity of gravitation along the path of light will be less than that of light, it will never overtake the light and the Doppler effect will be non-existent; only the intensity would diminish according to Newton's law

$$\text{Acceleration is } f = -\frac{GM}{r^2}\cdot\frac{\sqrt{r^2-a^2}}{r} + \frac{Gm}{(\sqrt{d^2-a^2}-R)^2} \quad \cdot \quad \cdot \quad \cdot \quad (39{\cdot}1)$$

$$\text{Hence } v^2 = v_0^2 + 2G.\int\left[\left\{-\frac{M}{r^2}\left(1-\frac{a^2}{2r^2}-\frac{1}{8}\frac{a^4}{r^4}\right)\right\} + \left\{\frac{m}{(\sqrt{d^2-a^2}-R)^2}\right\}\right]dR$$

$$= v^2{}_0 + 2G\left[\left\{\frac{M}{r} - \frac{Ma^2}{6 \cdot r^3} - \frac{a^4}{40.r^5}\right\} + \frac{m}{\sqrt{d^2 - a^2} - R}\right] \quad . \quad (39 \cdot 2)$$

At the Sun $v = v_s$, $r = a$ and $R = 0$. Choose $r_0 = c$ at ∞.

$$\therefore V^2{}_s = C^2 + 2\ G\left[\frac{M}{a}\left(1 - \tfrac{1}{6} - \tfrac{1}{40}\right) + \frac{m}{d\left(1 - \frac{a^2}{d^2}\right)^{\frac{1}{2}}}\right] \quad . \quad . \quad (39 \cdot 3)$$

At the Earth, $V = V_e$, $r = d - b$ and $R = \sqrt{d^2 - a^2} - b$

$$\therefore V^2{}_e = C^2 + 2\ G\left[\frac{M}{d-b}\left(1 - \tfrac{1}{6} - \tfrac{1}{40}\right) + \frac{m}{b}\right] \quad . \quad . \quad (39 \cdot 4)$$

As $\lambda = v.T$, it follows that if T is nearly the same for similar atoms, then $\lambda \propto v$.

$$\text{Hence } \frac{\lambda_s}{\lambda_e} = \frac{V_s}{V_e} = \frac{1 + \frac{G}{c^2}\left[\frac{97}{120} \cdot \frac{M}{a} + \frac{m}{d} \cdot \left(1 + \frac{1}{2}\frac{a^2}{d^2}\right)\right]}{1 + \frac{G}{c^2}\left[\frac{97}{120}\ \frac{M}{d-b} + \frac{m}{b}\right]} \quad . \quad . \quad . \quad (39 \cdot 5)$$

$$= \frac{97}{120} \times 1 \cdot 00000211 \text{ nearly}$$

$$= 1 \cdot 00677 \text{ for blue light} \quad . \quad . \quad . \quad . \quad . \quad . \quad (39 \cdot 6)$$

In such a case the displacement at the limb $= \cdot 00676$

3. If the path is hyperbolic, then obviously $V^2 = \frac{2\mu}{r} + \frac{\mu}{A}$. The spectral shift from the edge would then be nearly $\frac{\mu}{a}$, the same as that in the case of the light from the centre.

4. On Laplace's conception of the opposite velocity the spectral shift would be several times Enistein's value.

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9. Eddington : *Relativity*, p. 131.

10. Dated 11th May, 1935, p. 797.

11. Dated 30th November, 1934, p. 13, (also 16th November, 1934.)

12. December 1, 1934, (also March, 1935.)

13. Einstein : *Electrodynamics of Moving Bodies*, p. 4. See Chapter V, Sec. I, para , p. 241.

14. *Traite de Mecanique Celeste*, Tome IV, Chapter XIII, pp. 217—221.

15. Besant's *Dynamics*, Art. 157, pp. 207-8.

16. Routh's *Dynamics*, p. 98.

17. Tisserand : *Mecanique Celeste*, Ch. 29, p. 535.

NOTE.—A critic has objected to the accuracy of (9.4). The obvious explanation is that the terms $\frac{\mu^2}{h^4}$ and $\frac{\mu}{h^2}u$ were of the order 10^{-26} and were therefore neglected.

SECOND ERRATA

Chapter I

Page	Line	For	Read
13	18	$\frac{D^4}{\mu^3}$	$\frac{D^6}{\mu^3}$
"	22	"	"

Chapter V

Page	Line	For	Read
251	17	$+\frac{v^2}{D^2}$	$-\frac{v^2}{D^2}$
252	1	on	falling on
259	7	Eddington	Jeans

Chapter VIII

In Secs. I and II, pp. 148—152, $\frac{d^2r}{dt^2}$ and $\frac{d^2\theta}{dt^2}$ were for simplicity neglected. In reality they ought to be retained and the equations are more complex. The second method given on p. 152 is perhaps the more correct method, which will necessitate changes of sign in these sections.